Partially Palindromic Compositions

Jia Huang

University of Nebraska at Kearney
E-mail address: huangj2@unk.edu

Palindromic compositions

- A *composition* of n is a sequence $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ of positive integers with $\alpha_1 + \cdots + \alpha_\ell = n$; the *parts* of α are $\alpha_1, \ldots, \alpha_\ell$. There are 2^{n-1} compositions of n ($\leftrightarrow$ binary strings of length n ending with 1).

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- A composition $(\alpha_1, \dots, \alpha_\ell)$ of n is *palindromic* if $\alpha_i = \alpha_{\ell+1-i}$ for all $i = 1, \dots, \lfloor \ell/2 \rfloor$. The number of such compositions is $\mathrm{pc}(n) = 2^{\lfloor n/2 \rfloor}$.

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- A composition $(\alpha_1, \dots, \alpha_\ell)$ of n is *palindromic* if $\alpha_i = \alpha_{\ell+1-i}$ for all $i = 1, \dots, \lfloor \ell/2 \rfloor$. The number of such compositions is $\mathrm{pc}(n) = 2^{\lfloor n/2 \rfloor}$.
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- Just (2021+) defined a composition $\alpha = (\alpha_1, \dots, \alpha_\ell)$ to be *palindromic modulo m* if $\alpha_i \equiv \alpha_{\ell+1-i} \pmod m$ for all i and found the generating function for the number $\mathrm{pc}(n, m)$ of such compositions.

More on palindromic compositions

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- Andrews, Just, and Simay (2022) defined a composition $(\alpha_1, \ldots, \alpha_\ell)$ of n to be *anti-palindromic* if $\alpha_i \neq \alpha_{\ell+1-i}$ for all $i = 1, 2, \ldots, \lfloor \ell/2 \rfloor$ and showed that the number $\mathrm{ac}(n)$ of such compositions equals $T_n + T_{n-2}$, where T_n is a *tribonacci number* defined by $T_0 = 0$, $T_1 = T_2 = 1$, and $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for $n \geq 3$.

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- We view all compositions partially (anti-)palindromic (modulo m) and count them by the extent to which they are (anti-)palindromic.

Motivation

- A *partition* of n is a decreasing sequence $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ of positive integers with *size* $|\lambda| := \lambda_1 + \cdots + \lambda_\ell = n$ and *parts* $\lambda_1, \ldots, \lambda_\ell$.

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- There are parallel results on compositions: The number of compositions of n with odd parts and the number of compositions of n+1 with parts greater than one are both F_n. This was generalized by Munagi (2012) and further generalized by H. (2020).

Partially (anti-)palindromic compositions

- For $n, k \geq 0$, let $\mathrm{PC}^k(n)$ be the set of compositions $(\alpha_1, \ldots, \alpha_\ell)$ of n with $\#\{1 \leq i \leq \ell/2 : \alpha_i \neq \alpha_{\ell+1-i}\} = k$ and let $\mathrm{pc}^k(n) := |\mathrm{PC}^k(n)|$.

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- We have $\mathrm{pc}^k(n) = \mathrm{pc}^k_+(n) + \mathrm{pc}^k_-(n)$, where

$$\mathrm{pc}^k_+(n) := \#\{(\alpha_1, \ldots, \alpha_\ell) \in \mathrm{PC}^k(n) : 2 \mid \ell \text{ or } 2 \mid \alpha_{(\ell+1)/2}\},$$
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- Example: $\mathrm{pc}_+^1(4) = |\{31, 13\}| = 2$ and $\mathrm{pc}_-^1(4) = \{211, 112\}| = 2$.
- We define $\mathrm{ac}^k(n)$, $ac_+^k(n)$, and $\mathrm{ac}_-^k(n)$ similarly, using $\alpha_i = \alpha_{\ell+1-i}$ instead of $\alpha_i \neq \alpha_{\ell+1-i}$. We drop the superscript k if $k = 0$.

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- We have $\mathrm{pc}^k_+(n) = \mathrm{pc}^k_-(n+1)$, so $\mathrm{pc}^k(n) = \mathrm{pc}^k_+(n) + \mathrm{pc}^k_+(n-1)$, where $\mathrm{pc}^k_+(-1) := 0$; it is similar for $\mathrm{ac}^k(n)$.

Closed formulas for $\mathrm{pc}_+^k(n)$ and $\mathrm{ac}_+^k(n)$

- We show, both analytically and combinatorially, that

$$\mathrm{pc}_+^k(n) = \sum_{i+2j=n-3k} \binom{i+k-1}{i} \binom{j+k}{j} 2^{j+k},$$

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- We provide two more formulas for $\mathrm{ac}_+^k(n)$:

$$\mathrm{ac}_+^k(n) = \sum_{2r+i+j=n-2k} 2^i \binom{r+k}{k}\binom{r}{i}\binom{i+j-1}{j}$$

$$= \sum_{i+j+r+2s=n-2k} (-1)^i \binom{k+1}{i}\binom{j+k}{j}\binom{j}{r+s}\binom{r+s}{r}.$$

More on $\mathrm{pc}_+^k(n)$ and $\mathrm{ac}_+^k(n)$ for $k = 0, 1$

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- We have $\mathrm{pc}_+^1(n) = 2 + (\lceil n/2 \rceil - 2)2^{\lceil n/2 \rceil}$ [A036799].

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- The number $\mathrm{ac}_+(n)$ equals the *tribonacci number* T'_{n+1} with initial conditions $T'_0 = 0$, $T'_1 = 1$, $T'_2 = 0$ [A001590], so $\mathrm{ac}(n) = T'_{n+1} + T'_n$.

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- We provide another formula for $\mathrm{ac}^k(n)$:

$$\mathrm{ac}^k(n) = \sum_{i+j+r+s=n-2k} (-1)^i \binom{k}{i} \binom{j+k}{j} \binom{j}{r} \binom{r}{s} - \sum_{i+j+r+s=n-2k-2} (-1)^i \binom{k}{i} \binom{j+k}{j} \binom{j}{r} \binom{r}{s}.$$

This reduces to $\mathrm{ac}(n) = T_{n+1} - T_{n-1}$ when $k = 0$. As a byproduct, we get $T_{n+1} = \sum_{j+r+s=n} \binom{j}{r}\binom{r}{s}$, which has a simple bijective proof.

Reduced (anti-)palindromic compositions

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- Let $\mathrm{rpc}^k(n)$ (or $\mathrm{rac}^k(n)$) be the number of equivalence classes of compositions counted by $\mathrm{pc}^k(n)$ (or $\mathrm{ac}^k(n)$) under the above swaps. Define $\mathrm{rpc}^k_+(n)$, $\mathrm{rpc}^k_-(n)$, $\mathrm{rac}^k_+(n)$, and $\mathrm{rac}^k_-(n)$ similarly.

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- We have $\mathrm{rpc}_\pm^k(n) = \mathrm{pc}_\pm^k(n)/2^k$, so $\mathrm{rpc}^k(n) = \mathrm{pc}^k(n)/2^k$, and $\mathrm{rac}_+^k(n) = \sum_{2r+j=n-2k} \binom{r+k}{r}\binom{r+j-1}{j}$, which is also the number of compositions of $n-k$ with exactly k parts equal to 1 [A105422].

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- We have $\mathrm{rac}_+(n) = F_{n-1}$ and $\mathrm{rac}(n) = F_n$ for $n \geq 1$. and $\mathrm{rac}^1(n)$ counts compositions of $n-2$ with at most one even part [A208354].

Partially palindromic compositions modulo m

- Define $\mathrm{pc}^k(n, m)$ and $\mathrm{pc}^k_\pm(n, m)$ by replacing $\alpha_i \neq \alpha_{\ell+1-i}$ with $\alpha_i \not\equiv \alpha_{\ell+1-i} \pmod{m}$ in the definition of $\mathrm{pc}^k(n)$ and $\mathrm{pc}^k_\pm(n)$.

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- We provide two formulas for $\mathrm{pc}^k_+(n,m)$:

$$\mathrm{pc}^k_+(n,m) = \sum_{\substack{(m-1)r+s \\ =n-k-2i-mj}} (-1)^r 2^i \binom{i}{k}\binom{i+j-1}{j}\binom{k}{r}\binom{k+s-1}{s}$$

$$= \sum_{\substack{i_0+i_1+\cdots+i_{m-2}=k \\ i_1+2i_2+\cdots+(m-2)i_{m-2} \\ =n-k-2i-mj}} 2^i \binom{i}{k}\binom{i+j-1}{j}\binom{k}{i_0,i_1,\ldots,i_{m-2}}.$$

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- Taking $k=0$ gives $\mathrm{pc}_{+}(n,m) = \sum_{2i+mj=n} 2^i \binom{i+j-1}{j}$.

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- We have $\mathrm{pc}_+(n, 1) = \sum_{2i+j=n} 2^i \binom{i+j-1}{j}$ and $\mathrm{pc}_+^k(n, 1) = 0$ for $k \geq 1$. This sequence also counts compositions of n with parts greater than one, each part colored in two possible ways [A078008]. (Bijection?)

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- We have $\mathrm{pc}_+(n, 1) = \sum_{2i+j=n} 2^i \binom{i+j-1}{j}$ and $\mathrm{pc}_+^k(n, 1) = 0$ for $k \geq 1$. This sequence also counts compositions of n with parts greater than one, each part colored in two possible ways [A078008]. (Bijection?)
- We have $\mathrm{pc}(n, 1) = 2^{n-1}$ and $\mathrm{pc}^k(n, 1) = 0$ for $k \geq 1$.

More on $\mathrm{pc}_+(n, m)$ and $\mathrm{pc}(n, m)$

- Our formula for $\mathrm{pc}_+(n, m)$ implies some known results:
  - $\mathrm{pc}(2n, 2) = \mathrm{pc}(2n+1, 2) = 2 \cdot 3^{n-1}$ for $n \geq 1$,
  - $\mathrm{pc}(n, 3) = 2F_{n-1}$ for $n \geq 2$,
  - $\mathrm{pc}(2n, m) = \mathrm{pc}(2n+1, m)$ if m is even,
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- We have $\mathrm{pc}_+^k(n, 2) = \sum_{2i+2j=n-k} 2^i \binom{i}{k} \binom{i+j-1}{j}$, which is zero when $n-k$ is odd. In particular, for $n \geq 1$ we have $\mathrm{pc}_+^1(2n, 2) = 0$ and $\mathrm{pc}_+^1(2n+1, 2) = \sum_{i \geq 0} (i+1) 2^{i+1} \binom{n-1}{i}$ [A081038].

Reduced partially palindromic compositions modulo m

- Let $\mathrm{rpc}^k(n,m)$ be the number of equivalence classes of compositions counted by $\mathrm{pc}^k(n,m)$ under swaps of the ith part and the ith last part for all i. Define $\mathrm{rpc}^k_+(n,m)$ and $\mathrm{rpc}^k_-(n,m)$ similarly. We show

$$\begin{aligned}\mathrm{rpc}^k_+(n,m) &= \sum_{\substack{(m-1)r+s\\ =n-k-2i-mj-2c}} (-1)^r \binom{i}{k}\binom{i+j-1}{j}\binom{i+c}{c}\binom{k}{r}\binom{k+s-1}{s} \\ &= \sum_{\substack{i_0+i_1+\cdots+i_{m-2}=k\\ i_1+2i_2+\cdots+(m-2)i_{m-2}\\ =n-k-2i-mj-2c}} \binom{i}{k}\binom{i+j-1}{j}\binom{i+c}{c}\binom{k}{i_0,i_1,\ldots,i_{m-2}}.\end{aligned}$$

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- Taking $k=0$ gives $\mathrm{rpc}_+(n,m) = \sum_{2i+mj+2r=n} \binom{i+j-1}{j}\binom{i+r}{r}$.

More on $\mathrm{rpc}_+(n, m)$ for small k or m

- Taking $k = 0$ and $m = 1$ gives $\mathrm{rpc}_+(n, 1)$ [A052547] and $\mathrm{rpc}(n, 1) = \sum_{2i+j+2r=n} \binom{i+j}{j}\binom{i+r-1}{r}$ [A028495]; the latter also counts compositions of n with increments only appearing at every second position (such compositions are in bijection with the compositions counted by $\mathrm{rpc}(n, 1)$ by reordering parts appropriately).

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- We have $\mathrm{rpc}^k_+(n, 1) = 0$ and $\mathrm{rpc}^k_+(n, 2) = \sum_{2i+2j=n-k} \binom{i}{k}\binom{2i+j}{j}$. Thus $\mathrm{rpc}^1_+(2n, 2) = 0$, $\mathrm{rpc}^1_+(2n + 1, 2) = \sum_{0\le i\le n} i\binom{n+i}{2i}$ [A001870], and $\mathrm{rpc}^1(2n + 1, 2) = \mathrm{rpc}^1(2n + 2, 2) = \mathrm{rpc}^1_+(2n + 1, 2)$.

Partially anti-palindromic compositions modulo m

- We define $\mathrm{ac}^k(n,m)$, $\mathrm{ac}_+^k(n,m)$, and $\mathrm{ac}_-^k(n,m)$ by using $\equiv$ instead of $\not\equiv$ in the definition of $\mathrm{pc}^k(n,m)$, $\mathrm{pc}_+^k(n,m)$, and $\mathrm{pc}_-^k(n,m)$. We show

$$\begin{aligned}\mathrm{ac}_+^k(n,m) &= \sum_{\substack{2i+j+r(m-1)+s\\=n-2k-mc-md}} (-1)^r 2^j \binom{i+k}{k}\binom{i}{j}\binom{j}{r}\binom{j+s-1}{s}\binom{k}{c}\binom{k+j+d-1}{d}\\ &= \sum_{\substack{i_0+i_1+\cdots+i_{m-2}=j\\ i_1+2i_2+\cdots+(m-2)i_{m-2}\\=n-2k-2i-j-mc-md}} 2^j \binom{i+k}{k}\binom{i}{j}\binom{j}{i_0,\ldots,i_{m-2}}\binom{k}{c}\binom{k+j+d-1}{d}.\end{aligned}$$

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$$= \sum_{\substack{i_0+i_1+\cdots+i_{m-2}=j\\ i_1+2i_2+\cdots+(m-2)i_{m-2}\\=n-2k-2i-j-mc-md}} 2^j \binom{i+k}{k}\binom{i}{j}\binom{j}{i_0,\ldots,i_{m-2}}\binom{k}{c}\binom{k+j+d-1}{d}.$$

- We have another formula for $\mathrm{ac}^k(n,m)$:

$$\mathrm{ac}^k(n,m) = \sum_{\substack{3i+j+r(m-1)+2s\\=n-2k-mc-md}} (-1)^r 2^i \binom{i+k}{k}\binom{i+j}{j}\binom{i}{r}\binom{i+k+s-1}{s}\binom{k}{c}\binom{i+k+d-1}{d}.$$

More on $\mathrm{ac}^k(n,m)$ and $\mathrm{ac}_+^k(n,m)$ for small k or m

- Taking $k = 0$ gives $\mathrm{ac}_+(n,m)$ and $\mathrm{ac}(n,m)$, which are not in OEIS.

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- We have $\mathrm{ac}_+(n,1)=(1+(-1)^n)/2$ and for $k=1,2,3,4,5$ we find $\mathrm{ac}_+^k(n,1)$ in OEIS [A002620, A001752, A001769, A001780, A001786].

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- We have $\mathrm{ac}^k(n,1) = \binom{n}{2k}$, which counts compositions of n with $2k$ or $2k+1$ parts. A combinatorial proof: such compositions are in bijection with binary sequences of length n with exactly $2k$ ones (if the last digit is 0, change it to 1 to get a composition with $2k+1$ parts).

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- For $m=2,3$ or $k=0,1,2$ we cannot find $\mathrm{ac}^k(n,m)$ in OEIS.

Reduced partially anti-palindromic compositions modulo m

- Let $\mathrm{rac}^k(n, m)$ be the number of equivalence classes of compositions counted by $\mathrm{ac}^k(n, m)$ under swaps of the ith part and ith last part for all i. Define $\mathrm{rac}_+^k(n, m)$ and $\mathrm{rac}_-^k(n, m)$ similarly. We show that

$$\begin{aligned}\mathrm{rac}_+^k(n,m) &= \sum_{\substack{r(m-1)+s+md \\ =n-2k-2i-j}} (-1)^r \binom{i+k}{k}\binom{i}{j}\binom{j}{r}\binom{j+s-1}{s}\binom{k+j+d-1}{d} \\ &= \sum_{\substack{i_0+i_1+\cdots+i_{m-2}=j \\ i_1+2i_2+\cdots+(m-2)i_{m-2} \\ =n-2k-2i-j-md}} \binom{i+k}{k}\binom{i}{j}\binom{j}{i_0,\ldots,i_{m-2}}\binom{k+j+d-1}{d}.\end{aligned}$$

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- We have one more formula for $\mathrm{rac}^k(n, m)$:

$$\mathrm{rac}^k(n,m) = \sum_{\substack{r(m-1)+2s+dm \\ =n-2k-3i+j}} (-1)^r \binom{i+k}{k}\binom{i+j}{j}\binom{i}{r}\binom{i+k+s-1}{s}\binom{i+k+d-1}{d}.$$

More on $\mathrm{rac}_+^k(n, m)$ and $\mathrm{rac}^k(n, m)$ for $k = 0$ or $m = 1$

- For $n \geq 2$, the number $\mathrm{rac}_+(n, 2)$ counts compositions of $n - 2$ with no two adjacent parts of the same parity [A062200].

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- We have $\mathrm{rac}^k(n,1) = \sum_{2i+j=n-2k} \binom{i+k-1}{i}\binom{j+k}{j}$ [A060098]. Special cases include $\mathrm{rac}^1(n,1) = \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$ [A002620], $\mathrm{rac}^2(n,1)$ [A002624], $\mathrm{rac}^3(n,1)$ [A060099], $\mathrm{rac}^4(n,1)$ [A060100], and $\mathrm{rac}^5(n,1)$ [A060101].

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- Any combinatorial explanation for $\mathrm{rac}^1(n,1) = \mathrm{ac}_+^1(n,1)$?

Remarks and questions

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- Thank you very much for your attention!