Norton algebras of some distance regular graphs

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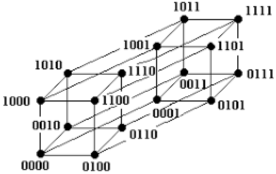
Distance regular graphs

- Let Γ be a *distance regular* graph with vertex set X, i.e., for any integers $i, j, k \geq 0$ and any pair $(x, y) \in X \times X$ with $d(x, y) = k$, the following *intersection number* does not depend on the choice of (x, y):
$$p_{ij}^k := \#\{z \in X : d(x, z) = i,\ d(y, z) = j\}.$$
- Suppose that Γ has *diameter* $d := \max\{d(x, y) : x, y \in X\}$. Then the adjacency matrix of Γ has distinct eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$, each of the same algebraic and geometric multiplicities.
- The vector space $\mathbb{R}^X := \{f : X \to \mathbb{R}\} \cong \mathbb{R}^{|X|}$ is a direct sum of the eigenspaces $V_0, V_1, \ldots, V_d$ of the eigenvalues $\theta_0, \theta_1, \ldots, \theta_d$.
- The eigenvalues and eigenspaces of Γ has many nice properties.

Hamming graphs

- The *Hamming graph* $H(n, e)$ has
  - vertex set $X = \{w_1 w_2 \cdots w_n : w_i \in \{0, 1, \ldots, e-1\}\}$ and
  - edge set $E = \{wu : w$ and u differ in precisely one position$\}$.
- The Hamming graph $H(n, 2) = Q_n$ is known as the *hypercube graph*.



- Two vertices have distance i iff they differ in precisely i positions.
- $H(n, e)$ is a distance regular graph of diameter $d = n$, whose ith eigenvalue is $\theta_i = (n-i)e - n$ with multiplicity $\dim(V_i) = \binom{n}{i}(e-1)^i$.
- The automorphism group of $H(n, e)$ is the wreath product $\mathfrak{S}_e \wr \mathfrak{S}_n$.

Norton algebra

- *Orthogonal projection* $\pi_i : \mathbb{R}^X = V_0 \oplus V_1 \oplus \cdots \oplus V_d \to V_i$.
- *Entry-wise product*: $(u \cdot v)(x) := u(x)v(x)$, $\forall u, v \in \mathbb{R}^X$, $\forall x \in X$.
- Define the *Norton product* on V_i by $u \star v := \pi_i(u \cdot v)$, $\forall u, v \in V_i$.
- The *Norton algebra* $(V_i, \star)$ is commutative but not associative.
- The Norton algebras have interesting automorphism groups and are related to the construction of the monster simple group.
- We determine the Norton algebras of certain distance regular graphs.
- We investigate the automorphism group of the Norton algebra.
- We also measure the nonassociativity of the Norton product $\star$.

Nonassociativity of binary operation

- Let $*$ be a binary operation on a set X. Let $x_0, x_1, \ldots, x_n$ be X-valued indeterminates.
- If $*$ is associative then the expression $x_0 * x_1 * \cdots * x_n$ is unambiguous. Example: $x_0 + x_1 + \cdots + x_n$.
- If $*$ is nonassociative then $x_0 * x_1 * \cdots * x_n$ depends on parentheses. The number of ways to parenthesize $x_0 * x_1 * \cdots * x_n$ is the *Catalan number* $C_n := \frac{1}{n+1}\binom{2n}{n}$, e.g., $(C_n)_{n=0}^{6} = (1, 1, 2, 5, 14, 42, 132)$.

$$\left.\begin{array}{l} ((x_0-x_1)-x_2)-x_3 = x_0 - x_1 - x_2 - x_3 \\ (x_0-x_1)-(x_2-x_3) = x_0 - x_1 - x_2 + x_3 \\ (x_0-(x_1-x_2))-x_3 = x_0 - x_1 + x_2 - x_3 \\ x_0-((x_1-x_2)-x_3) = x_0 - x_1 + x_2 + x_3 \\ x_0-(x_1-(x_2-x_3)) = x_0 - x_1 + x_2 - x_3 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} C_3 = 5 \\ C_{-,3} = 4 \\ \widetilde{C}_{-,3} = 2 \end{array}\right.$$

Nonassocitivity measurements

- Parentheses of $x_0 * x_1 * \cdots * x_n$ are *$*$-equivalent* if they give the same function from X^{n+1} to X. Let $C_{*,n}$ and $\widetilde{C}_{*,n}$ be the number of $*$-equivalence classes and the largest size of an $*$-equivalence class.
- In general, $1 \leq C_{*,n} \leq C_n$ and $1 \leq \widetilde{C}_{*,n} \leq C_n$. Moreover, we have $C_{*,n} = 1, \forall n \geq 0 \Leftrightarrow *$ is associative $\Leftrightarrow \widetilde{C}_{*,n} = C_n, \forall n \geq 0$. Thus $C_{*,n}$ and $\widetilde{C}_{*,n}$ measure how far $*$ is from being associative.
- Csákány and Waldhauser called $C_{*,n}$ the *associative spectrum* of $*$ while Braitt and Silberger called it the *subassociativity type*.
- Independently, we studied $C_{*,n}$ and $\widetilde{C}_{*,n}$ for a family of binary operations generalizing $+$ and $-$ (defined by using roots of unity and nilpotent elements in $\mathbb{C}[x,y]$) in joint work with Hein.
- Say $*$ is *totally nonassociative* if $C_{*,n} = C_n$ (or $\widetilde{C}_{*,n} = 1$), $\forall n \geq 0$.

Double Minus

Definition

- Define *double minus operation* $a \ominus b := -a - b$ for all $a, b \in \mathbb{R}$.
- Let $C_{\ominus,n,r}$ be the number of distinct results from $x_0 \ominus x_1 \ominus \cdots \ominus x_n$ with exactly r plus signs, so $C_{\ominus,n} := \sum_{0 \le r \le n+1} C_{\ominus,n,r}$.

Theorem (H., Mickey, and Xu 2017)

- *If $n \ge 1$ and $0 \le r \le n+1$ then*

$$C_{\ominus,n,r} = \begin{cases} \binom{n+1}{r}, & \text{if } n+r \equiv 1 \pmod 3 \text{ and } n \neq 2r-2, \\ \binom{n+1}{r} - 1, & \text{if } n+r \equiv 1 \pmod 3 \text{ and } n = 2r-2, \\ 0, & \text{if } n+r \not\equiv 1 \pmod 3. \end{cases}$$

- *For $n \ge 1$ we have* $C_{\ominus,n} = \begin{cases} \frac{2^{n+1}-1}{3}, & \text{if } n \text{ is odd;} \\ \frac{2^{n+1}-2}{3}, & \text{if } n \text{ is even.} \end{cases}$

A truncated/modified Pascal Triangle

Example ($C_{\ominus,n,r}$ for $n \leq 10$ and $0 \leq r \leq n+1$)

r	0	1	2	3	4	5	6	7	8	9	10	11
$C_{\ominus,0,r}$		1										
$C_{\ominus,1,r}$	1											
$C_{\ominus,2,r}$			2									
$C_{\ominus,3,r}$		4			1							
$C_{\ominus,4,r}$	1			9								
$C_{\ominus,5,r}$			15			6						
$C_{\ominus,6,r}$		7			34			1				
$C_{\ominus,7,r}$	1			56			28					
$C_{\ominus,8,r}$			36			125			9			
$C_{\ominus,9,r}$		10			210			120			1	
$C_{\ominus,10,r}$	1			165			461			55		

OEIS A000975

Definition

The sequence A000975 $(A_n : n \geq 1) = (1, 2, 5, 10, 21, 42, 85, \ldots)$ in OEIS has many equivalent characterizations, such as the following.

- $A_1 = 1$, $A_{n+1} = 2A_n$ if n is odd, and $A_{n+1} = 2A_n + 1$ if n is even.
- A_n is the integer with an alternating binary representation of length n. $(1 = 1_2,\ 2 = 10_2,\ 5 = 101_2,\ 10 = 1010_2,\ 21 = 10101_2,\ \ldots)$
- $A_n = \left\lfloor \frac{2^{n+1}}{3} \right\rfloor = \frac{2^{n+2}-3-(-1)^n}{6} = \begin{cases} \frac{2^{n+1}-1}{3}, & \text{if } n \text{ is odd;} \\ \frac{2^{n+1}-2}{3}, & \text{if } n \text{ is even.} \end{cases}$
- A_n is the number of moves to solve the n-ring Chinese Rings puzzle. $n = 4$: 0000-0001-0011-0010-0110-0111-0101-0100-1100-1101-1111

Question

- *Bijections between different objects enumerated by A_n?*
- *Any formula for $\widetilde{C}_{\ominus,n}$? $(1, 1, 1, 2, 3, 5, 9, 16, 28, 54, 99, \ldots)$*

Norton algebra of Hamming graph

Theorem (H. 2021)

- *Each eigenspace V_i of $H(n,e)$ has a basis $\{\tau_u : u \in X_i\}$, where X_i is the set of vertices with exactly i nonzero entries.*
- *If $u, v \in X_i$ then with $u+v$ defined component-wise modulo e,*
$$\tau_u \star \tau_v = \begin{cases} \tau_{u+v} & \text{if } u+v \in X_i \\ 0 & \text{otherwise.} \end{cases}$$
- *For $e \geq 3$, the automorphism group of $(V_i, \star)$ is trivial if $i = 0$, is isomorphic to $\mathfrak{S}_e \wr \mathfrak{S}_n$ if $i = 1$ or $\mathfrak{S}_3 \wr \mathfrak{S}_{2^{n-1}}$ if $i = n$ and $e = 3$, and admits a subgroup isomorphic to $(\mathbb{Z}_e \rtimes \mathbb{Z}_e^\times) \wr \mathfrak{S}_n$ if $i \geq 1$.*
- *The product $\star$ on V_i is associative if $i = 0$, equally as nonassociative as the double minus operation $\ominus$ if $e = 3$ and $i \in \{1, n\}$, or totally nonassociative if $e = 3$ and $1 < i < n$ or if $e \geq 4$ and $1 \leq i \leq n$.*

Examples: $H(2,3)$ and $H(3,2)$

Example ($H(2,3)$)

$\star$	τ_{01}	τ_{02}	τ_{10}	τ_{20}
τ_{01}	τ_{02}	0	0	0
τ_{02}	0	τ_{01}	0	0
τ_{10}	0	0	τ_{20}	0
τ_{20}	0	0	0	τ_{10}

$V_1(H(2,3))$

$\star$	τ_{11}	τ_{12}	τ_{21}	τ_{22}
τ_{11}	τ_{22}	0	0	0
τ_{12}	0	τ_{21}	0	0
τ_{21}	0	0	τ_{12}	0
τ_{22}	0	0	0	τ_{11}

$V_2(H(2,3))$

Example ($H(3,2)$)

The Norton algebra $V_2(H(3,2))$ has a basis $\{\tau_R, \tau_S, \tau_T\}$, where $R = \{1,2\}$, $S = \{1,3\}$, $T = \{2,3\}$. We have

$$\tau_R \star \tau_R = \tau_S \star \tau_S = \tau_T \star \tau_T = 0,$$

$$\tau_R \star \tau_S = \tau_T, \quad \tau_S \star \tau_T = \tau_R, \quad \text{and} \quad \tau_T \star \tau_R = \tau_S.$$

Norton algebra of the hypercube

Theorem (H. 2021)

- *Each eigenspace V_i of Q_n has a basis $\{\chi_S : S \subseteq [n],\ |S| = i\}$.*
- *If $S, T \subseteq [n]$ with $|S| = |T| = i$ then*

$$\chi_S \star \chi_T = \begin{cases} \chi_{S \triangle T} & \text{if } |S \triangle T| = i \\ 0 & \text{otherwise} \end{cases}$$

  where $S \triangle T := (S - T) \cup (T - S)$.
- *The automorphism group of $(V_i, \star)$ is trivial if $i = 0$, equals the general linear group of V_i if $i > \lfloor 2n/3 \rfloor$ or i is odd, and admits $\mathfrak{S}_n^B/\{\pm 1\}$ as a subgroup if $1 \le i < n$ is even ($\mathfrak{S}_n^B \cong \mathbb{Z}_2 \wr \mathfrak{S}_n$).*
- *The product $\star$ on V_i is associative if $i = 0$, $i > \lfloor 2n/3 \rfloor$ or i is odd, but totally nonassociative otherwise.*

Linear characters and Cayley graphs

- A *linear character* of a group G is a homomorphism $\chi : G \to \mathbb{C}^\times$ from G to the multiplicative group of nonzero complex numbers.
- The linear characters of G form an abelian group G^* under the *entry-wise product* defined by

$$(\chi \cdot \chi')(g) := \chi(g)\chi'(g) \quad \text{for all } \chi, \chi' \in G^* \text{ and } g \in G.$$

- Assume G is abelian. Then G^* is isomorphic to G and is an (orthonormal) basis for the space $\mathbb{C}^G := \{\phi : G \to \mathbb{C}\} \cong \mathbb{C}^{|G|}$.
- Let G be a finite abelian group expressed additively, and let S be a subset of $G - \{0\}$ such that $s \in S \Rightarrow -s \in S$.
- The *Cayley graph* $\Gamma(G, S)$ of G with respect to S has vertex set $X = G$ and edge set $E = \{xy : y - x \in S\}$.

Cayley graphs of finite abelian groups

Theorem (Well-known, see [Exercise 11.8, Lovasz 1979])

For any Cayley graph $\Gamma = \Gamma(X, S)$ of a finite abelian group X, the linear characters of X form an eigenbasis of Γ with each linear character χ corresponding to the eigenvalue $\chi(S) := \sum_{s \in S} \chi(s)$.
(This can be extended to the nonabelian case [Babai 1979, Lovász 1975].)

Theorem (H. 2021)

For any Cayley graph $\Gamma(X, S)$ of a finite abelian group X, we can define the Norton product $\chi \star \chi'$ of two linear characters χ and χ' in the same eigenspace by projecting the entry-wise product $\chi \cdot \chi'$ back to this eigenspace, and this product satisfies

$$\chi \star \chi' = \begin{cases} \chi \cdot \chi' & \text{if } (\chi \cdot \chi')(S) = \chi(S) \\ 0, & \text{otherwise.} \end{cases}$$

Other distance regular Cayley graphs

- The *folded cube* $\square_n$ can be obtained from Q_n by identifying each pair of vertices at distance n from each other.
- The *half-cube* $\frac{1}{2}Q_n$ can be obtained from the hypercube Q_n by selecting vertices with an even number of ones and drawing edges between pairs of vertices differing in exactly two positions.
- The *folded half-cube* $\frac{1}{2}\square_n$ is obtained from Q_n by folding and halving.
- The *bilinear forms graph* $H_q(d, e)$ has vertex set $X = \mathrm{Mat}_{d,e}(\mathbb{F}_q)$ consisting of all $d \times e$ matrices over a finite field $\mathbb{F}_q$ and has edge set E consisting of unordered pairs of $x, y \in X$ with $\mathrm{rank}(x - y) = 1$.
- Our linear character approach applies to the above distance regular graphs, as they are all Cayley graphs of finite abelian groups.

Johnson graphs

- The *Johnson graph* $J(n,k) = (X,E)$ has
  - vertex set $X = \{k\text{-subsets of } [n] := \{1,\ldots,n\}\}$ and
  - edge set $E = \{xy : x,y \in X,\ |x \cap y| = k-1\}$.
- For any $x, y \in X$, we have $d(x,y) = j$ if and only if $|x \cap y| = k - j$.
- We may assume $n \geq 2k$ since $J(n,k) \cong J(n,n-k)$ by taking set complement $(|x \cap y| = k-1 \Leftrightarrow |x^c \cap y^c| = n-k-1)$.
- The number of vertices at distance r from any vertex is $\binom{k}{r}\binom{n-k}{r}$.
- Thus $J(n,k)$ is a distance-regular graph with diameter $d = k$.
- $J(n,1)$ is the complete graph K_n and $J(n,2)$ is the line graph of K_n.
- The ith eigenvalue of $J(n,k)$ is $\theta_i = (k-i)(n-k-i) - i$ whose multiplicity is $\dim(V_i) = \binom{n}{i} - \binom{n}{i-1}$.

Grassmann graphs and dual polar graphs

- The *Grassmann graph* $J_q(n,k)$ is a q-analogue of $J(n,k)$, with
  - vertex set $X = \left\{k\text{-dimensional subspaces of } \mathbb{F}_q^n\right\}$, and
  - edge set $\{xy : x, y \in X,\ \dim(x \cap y) = k-1\}$.
- Given a vector space V with a quadratic/symplectic/Hermitian form, the *dual polar graph* Γ has
  - vertex set $X = \{\text{maximal isotropic subspaces of } V\}$, and
  - edge set $E = \{xy : x, y \in X,\ \dim(x \cap y) = d-1\}$, where $d := \dim(x)$, $\forall x \in X$ is well defined and is the diameter of Γ.
- The Grassmann graphs and dual polar graphs are distance regular.
- Levstein, Maldonado, and Penazzi (2009, 2012) determined the Norton algebra $(V_1, \star)$ of Johnson, Grassmann, and dual polar graphs.
- We measured the nonassociativity of $(V_1, \star)$ for these graphs (2020).

Lattice associated with distance regular graphs

Theorem (Levstein, Maldonado, and Penazzi, 2009, 2012)

Let $\Gamma = (X, E)$ be $J(n,d)$, $J_q(n,d)$, $H(d,2)$ or a dual polar graph of diameter d. There is a graded lattice $L = L_0 \sqcup L_1 \sqcup \cdots \sqcup L_{d+1}$ with $L_0 = \{\hat{0}\}$, $L_d = X$, $L_{d+1} = \hat{1}$, such that the following holds.
(i) There is a filtration $\Lambda_0 \subseteq \Lambda_1 \subseteq \cdots \subseteq \Lambda_d = \mathbb{R}^X$, where Λ_i is the span of the functions $\imath_v \in \mathbb{R}^X$ defined below for all $v \in L_i$:

$$\imath_v(x) := \begin{cases} 1 & \text{if } v \leq x \\ 0 & \text{otherwise.} \end{cases}$$

(ii) We have $V_0 = \Lambda_0 = \mathbb{R}\mathbf{1}$ and $V_i = \Lambda_i \cap \Lambda_{i-1}^{\perp}$ for $i = 1, 2, \ldots, d$.
(iii) The set $\{\check{v} : v \in L_1\}$ spans V_1, where $\check{v} := \pi_1(\imath_v) = \imath_v - \frac{a_1}{|X|}\mathbf{1}$ with $a_1 := \#\{x \in X : x \geq v\}$ not depending on the choice of v.

Remark

The above result is still valid for the Hamming graph $H(d,e)$.

Norton algebra $(V_1, \star)$

Theorem (Maldonado and Penazzi, 2012)

The eigenspace V_1 of the Johnson graph $J(n,k)$ has a spanning set $\{\check{v}_1, \ldots, \check{v}_n\}$ such that if $u, v \in L_1$ then

$$\check{u} \star \check{v} = \begin{cases} \left(1 - \frac{2k}{n}\right)\check{v} & \text{if } u = v \\ \frac{2k-n}{n(n-2)}(\check{u} + \check{v}) & \text{if } u \neq v. \end{cases}$$

Proposition (H. 2020)

- *If $n > 2k$ the Norton algebra $V_1(J(n,k))$ is isomorphic to $V_1(H(1,n))$.*
- *For $k \geq 2$ the Norton algebra $V_1(J_q(n,k))$ is totally nonassociative.*
- *The Norton algebra $(V_1, \star)$ of a dual polar graph Γ is totally nonassociative if $\Gamma \neq D_2(2)$ or equally as nonassociative as the double minus operation $\ominus$ if $\Gamma = D_2(2)$.*

Questions and remarks

- Terwilliger (2021) obtained nice formulas for the Norton algebra $(V_1, \star)$ of all *Q-polynomial* distance regular graphs. What about the Norton algebra $(V_i, \star)$ for $i > 1$?
- For a Cayley graph we can extend scalars to complex numbers and use linear characters, even though the adjacency matrix of any graph is a real symmetric matrix with real eigenvalues and eigenspaces.
- The eigenspaces of $J(n, k)$ can be constructed by linear algebra (Burcroff 2017) or representation theory (Krebs and Shaheen 2008)
- The Norton algebras we have studied so far are either associative, totally nonassociative, or equally as nonassociative as the double minus operation $\ominus$. Is there any intuitive explanation for this?

Thank you!