Variations of the Catalan number from nonassociative binary operations

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This is joint work with Nickolas Hein (Benedictine College), Madison Mickey (UNK) and Jianbai Xu (UNK)

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- $C_{*,n} = 1, \forall n \ge 0 \Leftrightarrow *$ is associative $\Leftrightarrow \widetilde{C}_{*,n} = C_n, \forall n \ge 0$.
- Thus $C_{*,n}$ and $\widetilde{C}_{*,n}$ measure how far $*$ is away from being associative.

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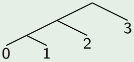

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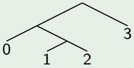

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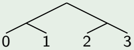

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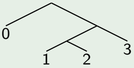

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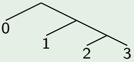

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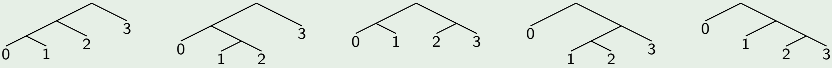

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Definition

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A generalization of associativity

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- A binary operation $*$ is *k-associative* if

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Let $\omega := e^{2\pi i/k}$ be a primitive kth root of unity. Then $*$ is k-associative if

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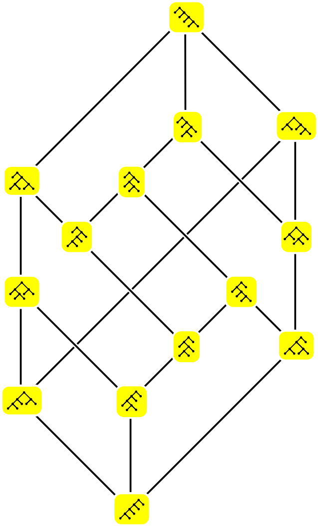
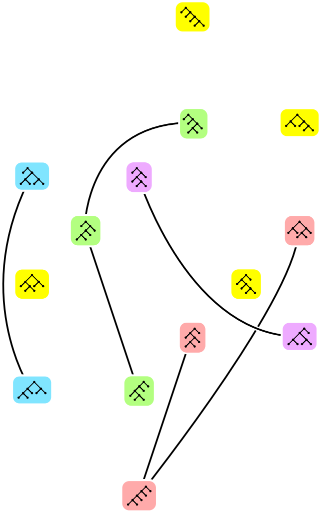
Observation (A generalization of the Tamari order)

The k-associativity gives the k-associative order on binary trees.

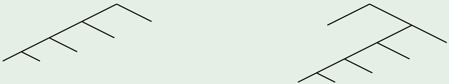
Tamari order and 2-associative order on $\mathcal{T}_4$





Components of k-associative order

Example (comb_4 and comb_4^1)



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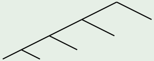

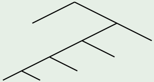

Theorem (Hein and H. 2017)

- *A binary tree is maximal (or minimal) in the k-associative order if and only if it avoids the binary tree comb_{k+1} (or comb_k^1) as a subtree.*

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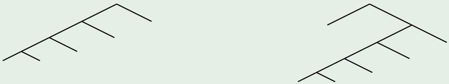

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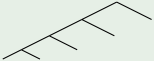

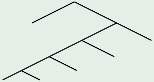

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Two binary trees t and t' correspond to equivalent parenthesizations if and only if $\delta_i(t) \equiv \delta_i(t') \pmod{k}$ for all i.

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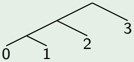

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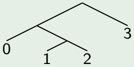

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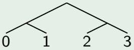

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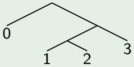

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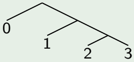

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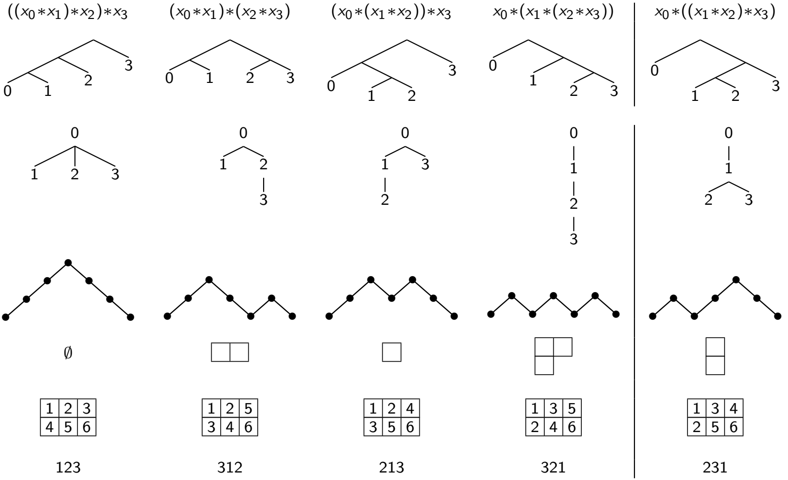
Proposition (Hein and H. 2017)

For $n \geq 0$ and $k \geq 1$, $C_{k,n}$ enumerates the following:

1. *the set of binary trees with $n+1$ leaves avoiding comb_k^1,*
2. *plane trees with n non-root nodes, each of degree less than k,*
3. *Dyck paths of length $2n$ avoiding DU^k (a down-step immediately followed by k up-steps),*
4. *partitions bounded by $(n-1, n-2, \ldots, 1, 0)$ with each positive part occurring fewer than k times,*
5. *$2 \times n$ standard Young tableaux which contain no list of k consecutive numbers in the top row other than $1, 2, \ldots, \ell$ for any $\ell \in [n]$,*
6. *permutations of $[n]$ avoiding 1-3-2 and $23 \cdots (k+1)1$.*

Examples of Catalan objects



The objects on each row are counted by the Catalan number C_3.
The rightmost column gives objects excluded by $C_{2,3}$.

Formulas for $C_{k,n}$ and $\widetilde{C}_{k,n}$

Theorem (Hein and H. 2017)

For $k, n \geq 1$, we have

$$C_{k,n} = \sum_{\substack{\lambda \subseteq (k-1)^n \\ |\lambda| < n}} \frac{n - |\lambda|}{n} m_\lambda(1^n) = \sum_{0 \leq j \leq (n-1)/k} \frac{(-1)^j}{n} \binom{n}{j} \binom{2n - jk}{n+1},$$

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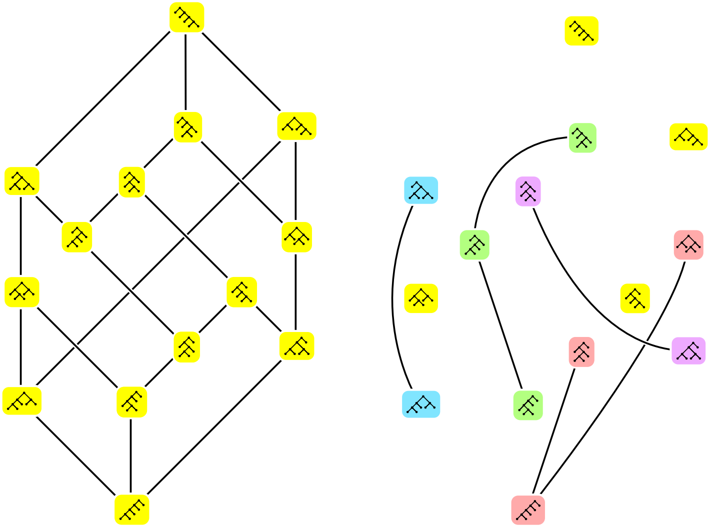
Proof.

One proof uses generating functions and Lagrange inversion. The other proof is more direct, using Dyck paths (and sign-reversing involutions). □

Tamari order and 2-associative order on $\mathcal{T}_4$



Modular Catalan numbers

Example ($C_{k,n}$ for $n \leq 10$ and $k \leq 8$)

n	0	1	2	3	4	5	6	7	8	9	10	
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Is there a generalization of this formula from $k = 3$ *to* $k \geq 4$*?*

Double Minus

Definition

- Define $a * b := \omega a + \eta b$ for $a, b \in \mathbb{C}$, where $\omega := e^{2\pi i/k}$ and $\eta := e^{2\pi i/\ell}$. When $k = \ell = 2$ this gives $a \ominus b := -a - b$.

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- Let $C_{\ominus,n,r}$ be the number of distinct results from $x_0 \ominus x_1 \ominus \cdots \ominus x_n$ with exactly r plus signs. Let $C_{\ominus,n} := \sum_{0 \le r \le n+1} C_{\ominus,n,r}$.

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Theorem (H., Mickey, and Xu 2017)

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A truncated/modified Pascal Triangle

Example ($C_{\ominus,n,r}$ for $n \leq 10$ and $0 \leq r \leq n+1$)

r	0	1	2	3	4	5	6	7	8	9	10	11
$C_{\ominus,0,r}$		1										
$C_{\ominus,1,r}$	1											
$C_{\ominus,2,r}$			2									
$C_{\ominus,3,r}$		4			1							
$C_{\ominus,4,r}$	1			9								
$C_{\ominus,5,r}$			15			6						
$C_{\ominus,6,r}$		7			34			1				
$C_{\ominus,7,r}$	1			56			28					
$C_{\ominus,8,r}$			36			125			9			
$C_{\ominus,9,r}$		10			210			120			1	
$C_{\ominus,10,r}$	1			165			461			55		

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- *For* $n \geq 1$ *we have* $C_{\ominus,n} = \begin{cases} \frac{2^{n+1}-1}{3}, & \text{if } n \text{ is odd;} \\ \frac{2^{n+1}-2}{3}, & \text{if } n \text{ is even.} \end{cases}$

OEIS A000975

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Question

- *Bijections between different objects enumerated by* A_n*?*
- *Any formula for* $\widetilde{C}_{\ominus,n}$*?* $(1, 1, 1, 2, 3, 5, 9, 16, 28, 54, 99, \ldots)$

Further Generalizations

- We can define $a * b := \omega a + \eta b$ for a, b in a ring R, where $\omega, \eta \in R$ satisfy $\omega^k = 1$ and $\eta^\ell = 1.$ But there is interference between ω and η.

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- Let $C_{k,\ell,n}^{d,e} := C_{*,n}$ and $\widetilde{C}_{k,\ell,n}^{d,e} := \widetilde{C}_{*,n}$ be, respectively, the number of equivalence classes and the largest size of an equivalence class of parenthesizations of $f_0 * f_1 * \cdots * f_n$.

The case $k = \ell = 1$: Associativity at depth (d, e)

Theorem (Hein and H. 2019+)

Let $k = \ell = 1$ and $t, t' \in \mathcal{T}_n$. Then $t \sim_* t'$ if and only if t be obtained from t' by a finite sequence of moves, each of which replaces the maximal subtree rooted at a node of left depth $\delta \geq d - 1$ and right depth $\rho \geq e - 1$ with a binary tree containing the same number of leaves.

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Theorem (Hein and H. 2019+)

- *If $n < d+e$ then $\widetilde{C}_n^{d,e} = 1$. If $n \geq d+e$ then $\widetilde{C}_n^{d,e} = n+2-d-e$ and the number of equivalence classes with this size is $\binom{d+e-2}{d-1}$.*

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- *The generating function $C^{d,e}(x) := \sum_{n\geq 0} C_n^{d,e} x^{n+1}$ satisfies*

$$C^{d,e}(x) = x + C^{d-1,e}(x) C^{d,e-1}(x)$$

where a zero in the supscript is treated as one.

The case $k = \ell = e = 1$

Corollary (Hein and H. 2019+)

The generating function $C^d(x) := C^{d,1}(x)$ *satisfies* $C^d(x) = \dfrac{x}{1 - C^{d-1}(x)}$.

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Example

$C^1(x) = \frac{x}{1-x}$, $C^2(x) = \frac{x}{1-\frac{x}{1-x}} = \frac{x(1-x)}{1-2x}$, $C^3(x) = \frac{x}{1-\frac{x}{1-\frac{x}{1-x}}} = \frac{x(1-2x)}{1-3x+x^2}$

n	1	2	3	4	5	6	7	n
C_n^1	1	1	1	1	1	1	1	1
C_n^2	1	2	4	8	16	32	64	2^{n-1}
C_n^3	1	2	5	13	34	89	233	F_{2n-1}
C_n^4	1	2	5	14	41	122	365	$\frac{1}{2}(1+3^{n-1})$
C_n	1	2	5	14	42	132	429	$\frac{1}{n+1}\binom{2n}{n}$

Some old results on C_n^d

Theorem (Kreweras 1970)

The number of Dyck paths of length $2n$ *with height at most* d *is* C_n^d *and*

$$C^d(x) = \frac{xF_{d+1}(x)}{F_{d+2}(x)}$$

where $F_i(x) := i$ *for* $i = 0, 1$, *and* $F_n(x) := F_{n-1}(x) - xF_{n-2}(x)$, $n \geq 2$.

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Theorem (de Bruijn–Knuth–Rice 1972)

The number of plane trees with $n+1$ nodes of depth at most d is

$$C_n^d = \frac{2^{2n+1}}{d+2} \sum_{1 \leq j \leq d+1} \sin^2(j\pi/(d+2)) \cos^{2n}(j\pi/(d+2)).$$

Moreover, $F_n(x) = \sum_{0 \leq i \leq (n-1)/2} \binom{n-1-i}{i} (-x)^i, \quad \forall n \geq 1.$

Recent results on C_n^d

Theorem (Andrews–Krattenthaler–Orsina–Papi 2002)

The number of ad-nilpotent ideals of the Borel subalgebra $\mathfrak{b}$ *of the Lie algebra* $\mathfrak{sl}_n(\mathbb{C})$ *with order at most* $d-1$ *is*

$$\begin{aligned} C_n^d &= \sum_{i\in\mathbb{Z}} \frac{2i(d+2)+1}{2n+1}\binom{2n+1}{n-i(d+2)} \\ &= \det\left[\binom{i-\max\{-1,j-d\}}{j-i+1}\right]_{i,j=1}^{n-1} \\ &= \sum_{0=i_0\le i_1\le\cdots\le i_{d-1}\le i_d=n}\ \prod_{0\le j\le d-2}\binom{i_{j+2}-i_j-1}{i_{j+1}-i_j}. \end{aligned}$$

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Theorem (Kitaev–Remmel–Tiefenbruck 2012)

The number of permutations in the symmetric group $\mathfrak{S}_n$ *avoiding* 132 *and* $123\cdots(d+1)$ *is* C_n^d.

New results on C_n^d

Definition

A composition of n is a sequence $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ of positive integers such that $\alpha_1 + \cdots + \alpha_\ell = n$. Let $\max(\alpha) := \max\{\alpha_1, \ldots, \alpha_\ell\}$ and $\ell(\alpha) = \ell$.

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For $n, d \geq 1$, the number C_n^d enumerates nilpotent ideals of the algebra $\mathcal{U}_n$ of n-by-n upper triangular matrices with order at most d.

Ideals of upper triangular matrices

Definition

- Let $\mathcal{U}_n$ be the algebra of all n-by-n upper triangular matrices

$$\begin{pmatrix} * & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ 0 & 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & * \end{pmatrix}$$

where a star $*$ is an arbitrary entry from a fixed field $\mathbb{F}$ (e.g., $\mathbb{R}$).

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Nilpotent ideals

Example (A nilpotent ideal of $\mathcal{U}_6$ and its corresponding Dyck path)

$$I = \begin{bmatrix} 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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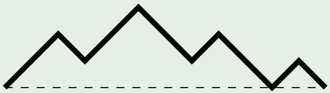

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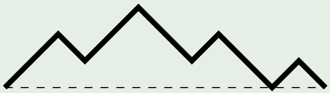

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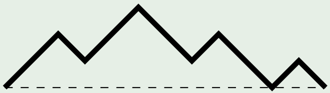

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Proposition (L. Shapiro, 1975)

The number of commutative ideals of $\mathcal{U}_n$ *is* $2^{n-1} (= C_n^2)$. *(Direct proof?)*

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Example

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Observation

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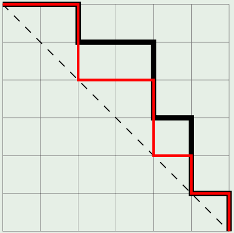

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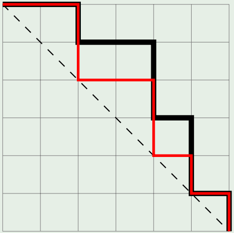

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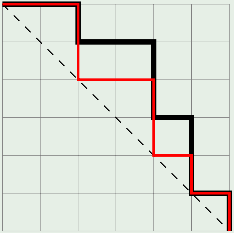

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Fact (Andrews–Krattenthaler–Orsina–Papi 2002, Haglund 2008)

Bijection ζ : Dyck paths with height d $\leftrightarrow$ Dyck paths with d bounces.

More on nilpotent ideals

Theorem (Hein and H. 2019+)

For $n, d \geq 1$, the number C_n^d enumerates nilpotent ideals of the algebra $\mathcal{U}_n$ of n-by-n upper triangular matrices with order at most d.

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- *The result on nilpotent ideas of $\mathfrak{b}$ has been generalized from type A to other types [Krattenthaler–Orsina–Papi 2002]. Is there a similar generalization for nilpotent ideals of $\mathcal{U}_n$?*

The case $e = \ell = 1$: k-associativity at left depth d

Theorem (Hein and H. 2019+)

We have $C^d_{2,n} = C^{d+1}_{1,n}$ *and for* $d, k \geq 1$ *and* $n \geq 0$,

$$C^d_{3,n} = \sum_{\substack{\alpha \models n+1 \\ h>1 \Rightarrow \alpha_h \leq d+1}} - \left(C^0_{3,\alpha_1-d-2} + \frac{\delta_{\alpha_1,d}}{2} + (-1)^{\alpha_1} \sum_{i+j=\alpha_1-1} \binom{d-i}{i}\binom{d+1-j}{j} \right) \cdot \prod_{h\geq 2} \left(\left(\delta_{\alpha_h,d} + (-1)^{\alpha_h-1} \sum_{i+j=\alpha_h} \binom{d+1-i}{i}\binom{d+1-j}{j} \right) \right)$$

$$C^2_{k,n} = 1 + \sum_{1\leq i\leq n-1} \frac{i}{n-i} \sum_{0\leq j\leq (n-i-1)/k} (-1)^j \binom{n-i}{j}\binom{2n-i-jk-1}{n}$$

$$= 1 + \sum_{1\leq i\leq n-1} \sum_{\lambda\subseteq (k-1)^{n-i}} \frac{n-i-|\lambda|}{n-i} \binom{n-|\lambda|-1}{n-|\lambda|-i} m_\lambda(1^{n-i}).$$

A final question

Conjecture

For $k, \ell \geq 1$ and $n \geq 0$ the equality $C_{k,\ell,n} = C_{k+\ell-1,n}$ holds.

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Thank you!