0-Hecke algebra actions on flags, polynomials, and Stanley-Reisner rings

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Dedication

I dedicate this thesis to my wife Ting Zou and our son Yifan.

Abstract

We study combinatorial aspects of the representation theory of the 0-Hecke algebra $H_n(0)$, a deformation of the group algebra of the symmetric group $\mathfrak{S}_n$.

We study the action of $H_n(0)$ on the polynomial ring in n variables. We show that the coinvariant algebra of this action naturally carries the regular representation of $H_n(0)$, giving an analogue of the well-known result for the symmetric group by Chevalley-Shephard-Todd.

By investigating the action of $H_n(0)$ on coinvariants and flag varieties, we interpret the generating functions counting the permutations with fixed inverse descent set by their inversion number and major index.

We also study the $H_n(0)$-action on the cohomology rings of the Springer fibers, and similarly interpret the (noncommutative) Hall-Littlewood symmetric functions indexed by hook shapes.

We generalize the last result from hooks to all compositions by defining an $H_n(0)$-action on the Stanley-Reisner ring of the Boolean algebra. By studying this action we obtain a family of multivariate noncommutative symmetric functions, which specialize to the noncommutative Hall-Littlewood symmetric functions and their (q,t)-analogues introduced by Bergeron and Zabrocki, and to a more general family of noncommutative symmetric functions having parameters associated with paths in binary trees introduced recently by Lascoux, Novelli, and Thibon.

We also obtain multivariate quasisymmetric function identities from this $H_n(0)$-action, which specialize to results of Garsia and Gessel on generating functions of multivariate distributions of permutation statistics.

More generally, for any finite Coxeter group W, we define an action of its Hecke algebra $H_W(q)$ on the Stanley-Reisner ring of its Coxeter complex. We find the invariant algebra of this action, and show that the coinvariant algebra of this action is isomorphic to the regular representation of $H_W(q)$ if q is generic. When $q = 0$ we find a decomposition for the coinvariant algebra as a multigraded $H_W(0)$-module.

Contents

Chapter 1

Introduction

1.1 Background and objective

The *symmetric group* $\mathfrak{S}_n$ consists of all permutations of $[n] := \{1, \ldots, n\}$ and has group operation being the composition of permutations. It is generated by $s_1, \ldots, s_{n-1}$, where $s_i := (i, i+1)$ is the transposition of i and $i+1$, with the following relations

$$\begin{cases} s_i^2 = 1, \ 1 \leq i \leq n-1, \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \ 1 \leq i \leq n-2, \\ s_i s_j = s_j s_i, \ |i-j| > 1. \end{cases}$$

The 0-*Hecke algebra* $H_n(0)$ can be obtained by deforming the above relations, that is, $H_n(0)$ is the algebra over an arbitrary field $\mathbb{F}$, generated by $\pi_1, \ldots, \pi_{n-1}$ with relations

$$\begin{cases} \pi_i^2 = \pi_i, \ 1 \leq i \leq n-1, \\ \pi_i \pi_{i+1} \pi_i = \pi_{i+1} \pi_i \pi_{i+1}, \ 1 \leq i \leq n-2, \\ \pi_i \pi_j = \pi_j \pi_i, \ |i-j| > 1. \end{cases}$$

In fact, the group algebra of $\mathfrak{S}_n$ and the 0-Hecke algebra $H_n(0)$ are specializations of the *Hecke algebra* $H_n(q)$ at $q = 1$ and $q = 0$, respectively (see §2.4).

One can also realize the symmetric group $\mathfrak{S}_n$ as a group of operators on sequences $a_1 \cdots a_n$ of n integers, with s_i swapping a_i and a_{i+1}. Similarly the 0-Hecke algebra $H_n(0)$ can be viewed as an algebra of operators on the same kind of sequences, with π_i

being the *bubble sorting operator* which swaps a_i and a_{i+1} if $a_i > a_{i+1}$ and does nothing otherwise. In order to sort an arbitrary sequence $a_1 \cdots a_n$ into an increasing sequence, it is sufficient to repeatedly applying $\pi_1, \ldots, \pi_{n-1}$. This sorting procedure is called the *bubble sort*, a well known algorithm in computer science.

The symmetric group naturally arises in many areas of mathematics as well as other subjects like physics, chemistry, etc. The structures and representations of the symmetric group have been widely studied and its applications have been found in various fields. The history dates back hundreds of years ago, and the results obtained and the methods employed have stimulated the development of many mathematical branches.

The 0-Hecke algebra, however, had not received much attention until recently. It was first studied by P.N. Norton around 1978 in her doctoral dissertation and her later journal paper [49]. We review her main results below.

A *composition* [*weak composition* resp.] of n is a sequence $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ of positive [nonnegative resp.] integers such that the size $|\alpha| := \alpha_1 + \cdots + \alpha_\ell$ of α equals n. The *descent (multi)set* of a (weak) composition α is the (multi)set

$$D(\alpha) := \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_{\ell-1}\}.$$

The map $\alpha \mapsto D(\alpha)$ is a bijection between (weak) compositions of n and (multi)sets with elements in $[n-1]$.

Norton [49] showed that

$$H_n(0) = \bigoplus_{\alpha \models n} \mathbf{P}_\alpha \tag{1.1}$$

summed over all compositions α of n, where every $\mathbf{P}_\alpha$ is a (left) indecomposable $H_n(0)$-module. It follows that $\{\mathbf{P}_\alpha : \alpha \models n\}$ is a complete list of pairwise non-isomorphic projective indecomposable $H_n(0)$-modules, and $\{\mathbf{C}_\alpha : \alpha \models n\}$ is a complete list of pairwise non-isomorphic simple $H_n(0)$-modules, where $\mathbf{C}_\alpha := \mathrm{top}(\mathbf{P}_\alpha) = \mathbf{P}_\alpha / \operatorname{rad} \mathbf{P}_\alpha$ for all compositions $\alpha \models n$. Norton's results inspired further studies on the 0-Hecke algebra by C.W. Carter [14], M. Fayers [21], P. McNamara [48], and so on.

When $\mathbb{F}$ is an algebraically closed field of characteristic zero, it is well known that the simple $\mathfrak{S}_n$-modules S_λ are indexed by partitions $\lambda \vdash n$, and every finite dimensional $\mathfrak{S}_n$-module is a direct sum of simple $\mathfrak{S}_n$-modules. In this case there is a classic correspondence between finite dimensional $\mathfrak{S}_n$-representations and the *ring of symmetric* *functions*, via the *Frobenius characteristic map* ch which sends a direct sum of simple $\mathfrak{S}_n$-modules to the sum of the corresponding Schur functions s_λ.

D. Krob and J.-Y. Thibon [39] provided an analogous correspondence for $H_n(0)$-representations (over an arbitrary field $\mathbb{F}$). The finite dimensional $H_n(0)$-modules correspond to the quasisymmetric functions via the *quasisymmetric characteristic* Ch which sends a simple $\mathbf{C}_\alpha$ to the *fundamental quasisymmetric function* F_α; the finite dimensional projective $H_n(0)$-modules correspond to the noncommutative symmetric functions via the *noncommutative characteristic* **ch** which sends a projective indecomposable $\mathbf{P}_\alpha$ to the *noncommutative ribbon Schur function* $\mathbf{s}_\alpha$. There are also graded versions of the two characteristic maps Ch and **ch** for $H_n(0)$-modules with gradings and filtrations.

The main objective of my research is to find circumstances where the 0-Hecke algebra naturally acts, and in such circumstances, study the resulting representations of the 0-Hecke algebra and compare them with the possible counterpart for the symmetric group. This is motivated by the aforementioned analogy between the 0-Hecke algebra and the symmetric group, as it is natural to ask for other evidence to support this analogy. We will focus on various representations of the symmetric group, and try to obtain similar results for the 0-Hecke algebra.

1.2 Coinvariants and flags

A composition α of n gives rise to a descent class of permutations in $\mathfrak{S}_n$; the cardinality of this descent class is known as the *ribbon number* r_α and its inv-generating function is the *q-ribbon number* $r_\alpha(q)$. Reiner and Stanton [50] defined a (q,t)*-ribbon number* $r_\alpha(q,t)$, and gave an interpretation by representations of the symmetric group $\mathfrak{S}_n$ and the finite general linear group $GL(n,\mathbb{F}_q)$. The goal of Chapter 3 is to find similar interpretations of various ribbon numbers by representations of the 0-Hecke algebra $H_n(0)$. The main results are summarized below.

The symmetric group $\mathfrak{S}_n$ acts on the polynomial ring $\mathbb{F}[X] := \mathbb{F}[x_1,\ldots,x_n]$ by permuting the variables, and the 0-Hecke algebra $H_n(0)$ acts on $\mathbb{F}[X]$ via the *Demazure operators* $\pi_1,\ldots,\pi_{n-1}$, where

$$\pi_i f := \frac{x_i f - x_{i+1} s_i(f)}{x_i - x_{i+1}}, \quad \forall f \in \mathbb{F}[X]. \tag{1.2}$$

The coinvariant algebra of $H_n(0)$ coincides with that of $\mathfrak{S}_n$, since $\pi_i f = f$ if and only if $s_i f = f$ for all $f \in \mathbb{F}[X]$. The following result for $\mathfrak{S}_n$ is well-known.

Theorem 1.2.1 (Chevalley-Shephard-Todd). *The coinvariant algebra* $\mathbb{F}[X]/(\mathbb{F}[X]_+^{\mathfrak{S}_n})$ *is isomorphic to the regular representation of* $\mathfrak{S}_n$*, if* $\mathbb{F}$ *is a field with* $\mathrm{char}(\mathbb{F}) = 0$.

We give the following analogous result for the 0-Hecke algebra.

Theorem 1.2.2. *The coinvariant algebra* $\mathbb{F}[X]/(\mathbb{F}[X]_+^{\mathfrak{S}_n})$ *with the* $H_n(0)$*-action defined by (1.2) is isomorphic to the regular representation of* $H_n(0)$ *for any field* $\mathbb{F}$.

We prove this theorem by showing a decomposition of the coinvariant algebra similar to Norton's decomposition (1.1). This leads to an $\mathbb{F}$-basis of the coinvariant algebra, which is closely related to the well-known basis of *descent monomials*. Our new basis consists of certain *Demazure atoms* obtained by consecutively applying the operators $\overline{\pi}_i = \pi_i - 1$ to some descent monomials. Theorem 1.2.2 and its proof are also valid when $\mathbb{F}$ is replaced with $\mathbb{Z}$.

It follows from Theorem 1.2.2 that the coinvariant algebra has not only the grading by the degrees of polynomials, but also the filtration by the length of permutations in $\mathfrak{S}_n$. This completes the following picture.

$$\begin{tikzcd}
& \boxed{\mathrm{Ch}(H_n(0)) \overset{\text{Krob-Thibon}}{=\!=\!=} \displaystyle\sum_{w\in\mathfrak{S}_n} F_{D(w^{-1})}} & \\
\boxed{\begin{array}{c}\mathrm{Ch}_q(H_n(0)) \overset{\text{Krob-Thibon}}{=\!=\!=} \\ \displaystyle\sum_{w\in\mathfrak{S}_n} q^{\mathrm{inv}(w)} F_{D(w^{-1})}\end{array}} \arrow[ur, squiggly, "q\to 1"] \arrow[rr, leftrightarrow, "q\leftrightarrow t", "\text{(Foata-Schützenberger)}"'] & & \boxed{\begin{array}{c}\mathrm{Ch}_t(\mathbb{F}[X]/(\mathbb{F}[X]_+^{\mathfrak{S}_n})) \overset{\text{New}}{=\!=\!=} \\ \displaystyle\sum_{w\in\mathfrak{S}_n} t^{\mathrm{maj}(w)} F_{D(w^{-1})}\end{array}} \arrow[ul, squiggly, "t\to 1"'] \\
& \boxed{\begin{array}{c}\mathrm{Ch}_{q,t}\left(\mathbb{F}[X]/(\mathbb{F}[X]_+^{\mathfrak{S}_n})\right) \overset{\text{New}}{=\!=\!=} \\ \displaystyle\sum_{w\in\mathfrak{S}_n} t^{\mathrm{maj}(w)} q^{\mathrm{inv}(w)} F_{D(w^{-1})}\end{array}} \arrow[ul, squiggly, "t\to 1"] \arrow[ur, squiggly, "q\to 1"'] &
\end{tikzcd}$$

Here the inverse descent set

$$D(w^{-1}) := \{i \in [n-1] : w^{-1}(i) > w^{-1}(i+1)\}$$

is identified with the composition α of n with $D(\alpha) = D(w^{-1})$. We will see that r_α and $r_\alpha(q)$ appear as coefficients of F_α in $\mathrm{Ch}(H_n(0))$ and $\mathrm{Ch}_q(H_n(0))$, respectively, in §2.7.

Next we consider the finite general linear group $G = GL(n, \mathbb{F}_q)$, where q is a power of a prime p, and its Borel subgroup B. The 0-Hecke algebra $H_n(0)$ acts on the flag variety $1_B^G = \mathbb{F}[G/B]$ by $T_w B = BwB$ if $\mathrm{char}(\mathbb{F}) = p$, and this induces an action on the coinvariant algebra $\mathbb{F}[X]^B/(\mathbb{F}[X]_+^G)$ of the pair (G, B) (see §3.3.3). By studying the (graded) multiplicities of the simple $H_n(0)$-modules in these $H_n(0)$-modules, we complete the following diagram, which interprets all the ribbon numbers mentioned earlier.

$$
\begin{tikzcd}
& \boxed{r_\alpha \xlongequal{\text{Krob-Thibon}} \mathrm{mult}_{\mathbf{C}_\alpha}(H_n(0))} & \\
\boxed{\begin{array}{c} r_\alpha(q) \xlongequal{\text{New}} \\ \mathrm{mult}_{\mathbf{C}_\alpha}(1_B^G) \end{array}} \arrow[ur, squiggly, "q\to 1"] \arrow[rr, leftrightarrow, "q \leftrightarrow t", "\text{(Foata-Schützenberger)}"'] & & \boxed{\begin{array}{c} r_\alpha(t) \xlongequal{\text{New}} \\ \mathrm{grmult}_{\mathbf{C}_\alpha}\left(\mathbb{F}[X]/(\mathbb{F}[X]_+^{\mathfrak{S}_n})\right) \end{array}} \arrow[ul, squiggly, "t\to 1"'] \\
& \boxed{\begin{array}{c} r_\alpha(q,t) \xlongequal{\text{New}} \\ \mathrm{grmult}_{\mathbf{C}_\alpha}\left(\mathbb{F}[X]^B/(\mathbb{F}[X]_+^G)\right) \end{array}} \arrow[ul, squiggly, "t\to 1"] \arrow[ur, squiggly, "{t \to t^{\frac{1}{q-1}},\ q \to 1}"'] &
\end{tikzcd}
$$

Finally we consider a family of quotient rings $R_\mu = \mathbb{F}[X]/J_\mu$ indexed by partitions μ of n, which contains the coinvariant algebra of $\mathfrak{S}_n$ as a special case ($\mu = 1^n$). If $\mathbb{F} = \mathbb{C}$ then R_μ is isomorphic to the cohomology rings of the Springer fiber $\mathcal{F}_\mu$, carries an $\mathfrak{S}_n$-action, and has graded Frobenius characteristic equal to the modified Hall-Littlewood symmetric function $\widetilde{H}_\mu(x;t)$ (see e.g. Hotta-Springer [33] and Garsia-Procesi [26]).

Using an analogue of the nabla operator, Bergeron and Zabrocki [8] introduced a noncommutative modified Hall-Littlewood symmetric function $\widetilde{\mathbf{H}}_\alpha(\mathbf{x};t)$ and a (q,t)-analogue $\widetilde{\mathbf{H}}_\alpha(\mathbf{x};q,t)$ for all compositions α.

We prove that the $H_n(0)$-action on $\mathbb{F}[X]$ preserves the ideal J_μ if and only if μ is a hook, and if so then R_μ becomes a projective $H_n(0)$-module whose graded characteristic also equals $\widetilde{H}_\mu(x;t)$, and whose graded noncommutative characteristic equals $\widetilde{\mathbf{H}}_\mu(\mathbf{x};t)$, where μ is viewed as a composition $(1^k, n-k)$ with increasing parts.

1.3 Stanley-Reisner ring of the Boolean algebra

Let $\mathbb{F}$ be an arbitrary field. The symmetric group $\mathfrak{S}_n$ naturally acts on the polynomial ring $\mathbb{F}[X] := \mathbb{F}[x_1, \ldots, x_n]$ by permuting the variables $x_1, \ldots, x_n$. The invariant algebra $\mathbb{F}[X]^{\mathfrak{S}_n}$, which consists of all the polynomials fixed by this $\mathfrak{S}_n$-action, is a polynomial algebra generated by the elementary symmetric functions $e_1, \ldots, e_n$. The coinvariant algebra $\mathbb{F}[X]/(\mathbb{F}[X]^{\mathfrak{S}_n}_+)$, with $(\mathbb{F}[X]^{\mathfrak{S}_n}_+) = (e_1, \ldots, e_n)$, is a vector space of dimension $n!$ over $\mathbb{F}$, and if n is not divisible by the characteristic of $\mathbb{F}$ then it carries the regular representation of $\mathfrak{S}_n$. A well known basis for $\mathbb{F}[X]/(\mathbb{F}[X]^{\mathfrak{S}_n}_+)$ consists of the descent monomials. Garsia [24] obtained this basis by transferring a natural basis from the Stanley-Reisner ring $\mathbb{F}[\mathcal{B}_n]$ of the Boolean algebra $\mathcal{B}_n$ to the polynomial ring $\mathbb{F}[X]$. Here the Boolean algebra $\mathcal{B}_n$ is the set of all subsets of $[n] := \{1, \ldots, n\}$ partially ordered by inclusion, and $\mathbb{F}[\mathcal{B}_n]$ is the quotient of the polynomial algebra $\mathbb{F}[y_A : A \subseteq [n]]$ by the ideal $(y_A y_B :\ A$ and B are incomparable in $\mathcal{B}_n)$.

The 0-Hecke algebra $H_n(0)$ acts on $\mathbb{F}[X]$ by the Demazure operators, having the same invariant algebra as the $\mathfrak{S}_n$-action on $\mathbb{F}[X]$. According to Theorem 1.2.2, the coinvariant algebra $\mathbb{F}[X]/(\mathbb{F}[X]^{\mathfrak{S}_n}_+)$ is also isomorphic to the regular representation of $H_n(0)$, for any field $\mathbb{F}$. We prove this result by constructing another basis for $\mathbb{F}[X]/(\mathbb{F}[X]^{\mathfrak{S}_n}_+)$ which consists of certain polynomials whose leading terms are the descent monomials. This and the previously mentioned connection between the Stanley-Reisner ring $\mathbb{F}[\mathcal{B}_n]$ and the polynomial ring $\mathbb{F}[X]$ motivate us to define an $H_n(0)$-action on $\mathbb{F}[\mathcal{B}_n]$.

We define such an action and investigate it in Chapter 4. It turns out that our $H_n(0)$-action on $\mathbb{F}[\mathcal{B}_n]$ has similar definition and properties to the $H_n(0)$-action on $\mathbb{F}[X]$. It preserves the $\mathbb{N}^{n+1}$-multigrading of $\mathbb{F}[\mathcal{B}_n]$ and has invariant algebra equal to a polynomial algebra $\mathbb{F}[\Theta]$, where Θ is the set of *rank polynomials* θ_i (the usual analogue of e_i in $\mathbb{F}[\mathcal{B}_n]$). We show that the $H_n(0)$-action is Θ-linear and thus descends to the coinvariant algebra $\mathbb{F}[\mathcal{B}_n]/(\Theta)$. We will see that $\mathbb{F}[\mathcal{B}_n]/(\Theta)$ carries the regular representation of $H_n(0)$.

Furthermore, using the $H_n(0)$-action on $\mathbb{F}[\mathcal{B}_n]$ we obtain a complete noncommutative analogue for the remarkable representation theoretic interpretation of the Hall-Littlewood symmetric functions mentioned in the previous section. In the analogous case of $H_n(0)$ acting on $\mathbb{F}[X]$, one only has a partial noncommutative analogue for hooks.

Theorem 1.3.1. *Let α be a composition of n. Then there exists a homogeneous $H_n(0)$-stable ideal I_α of the multigraded algebra $\mathbb{F}[\mathcal{B}_n]$ such that the quotient algebra $\mathbb{F}[\mathcal{B}_n]/I_\alpha$ becomes a projective $H_n(0)$-module with multigraded noncommutative characteristic equal to*

$$\widetilde{\mathbf{H}}_\alpha(\mathbf{x}; t_1, \ldots, t_{n-1}) := \sum_{\beta \preccurlyeq \alpha} \underline{t}^{D(\beta)} \mathbf{s}_\beta \quad \text{inside} \quad \mathbf{NSym}[t_1, \ldots, t_{n-1}].$$

Moreover, one has $\widetilde{\mathbf{H}}_\alpha(\mathbf{x}; t, t^2, \ldots, t^{n-1}) = \widetilde{\mathbf{H}}_\alpha(\mathbf{x}; t)$, and one obtains $\widetilde{\mathbf{H}}_\alpha(\mathbf{x}; q, t)$ from $\widetilde{\mathbf{H}}_{1^n}(\mathbf{x}; t_1, \ldots, t_{n-1})$ by taking $t_i = t^i$ for all $i \in D(\alpha)$, and taking $t_i = q^{n-i}$ for all $i \in [n-1] \setminus D(\alpha)$.

Here the notation $\beta \preccurlyeq \alpha$ means α and β are compositions of n with $D(\beta) \subseteq D(\alpha)$, and $\underline{t}^S$ denotes the product $\prod_{i \in S} t_i$ over all elements i in a multiset S, including the repeated ones. We also generalize the basic properties of $\widetilde{\mathbf{H}}_\alpha(\mathbf{x}; t)$ given in [8] to the multivariate $\widetilde{\mathbf{H}}_\alpha(\mathbf{x}; t_1, \ldots, t_{n-1})$.

Note that $\widetilde{\mathbf{H}}_{1^n}(\mathbf{x}; t_1, \ldots, t_{n-1})$ is the multigraded noncommutative characteristic of the coinvariant algebra $\mathbb{F}[\mathcal{B}_n]/(\Theta)$, from which one sees that $\mathbb{F}[\mathcal{B}_n]/(\Theta)$ carries the regular representation of $H_n(0)$. Specializations of $\widetilde{\mathbf{H}}_{1^n}(\mathbf{x}; t_1, \ldots, t_{n-1})$ include not only $\widetilde{\mathbf{H}}_\alpha(\mathbf{x}; q, t)$, but also a more general family of noncommutative symmetric functions depending on parameters associated with paths in binary trees introduced recently by Lascoux, Novelli, and Thibon [42].

Next we study the quasisymmetric characteristic of $\mathbb{F}[\mathcal{B}_n]$. We combine the usual $\mathbb{N}^{n+1}$-multigrading of $\mathbb{F}[\mathcal{B}_n]$ (recorded by $\underline{t} := t_1, \ldots, t_{n-1}$) with the length filtration of $H_n(0)$ (recorded by q) and obtain an $\mathbb{N} \times \mathbb{N}^{n+1}$-multigraded quasisymmetric characteristic for $\mathbb{F}[\mathcal{B}_n]$.

Theorem 1.3.2. *The $\mathbb{N} \times \mathbb{N}^{n+1}$-multigraded quasisymmetric characteristic of $\mathbb{F}[\mathcal{B}_n]$ is*

$$\begin{aligned} \mathrm{Ch}_{q,\underline{t}}(\mathbb{F}[\mathcal{B}_n]) &= \sum_{k \geq 0} \sum_{\alpha \in \mathrm{Com}(n,k+1)} \underline{t}^{D(\alpha)} \sum_{w \in \mathfrak{S}^\alpha} q^{\mathrm{inv}(w)} F_{D(w^{-1})} \\ &= \sum_{w \in \mathfrak{S}_n} \frac{q^{\mathrm{inv}(w)} \underline{t}^{D(w)} F_{D(w^{-1})}}{\prod_{0 \leq i \leq n} (1 - t_i)} \\ &= \sum_{k \geq 0} \sum_{\mathbf{p} \in [k+1]^n} t_{p'_1} \cdots t_{p'_k} q^{\mathrm{inv}(\mathbf{p})} F_{D(\mathbf{p})}. \end{aligned}$$

Here we identify F_I with F_α if $D(\alpha) = I \subseteq [n-1]$. The set $\mathrm{Com}(n,k)$ consists of all weak compositions of n with length k. We also define $\mathfrak{S}^\alpha := \{w \in \mathfrak{S}_n : D(w) \subseteq D(\alpha)\}$. The set $[k+1]^n$ consists of all words of length n on the alphabet $[k+1]$. Given $\mathbf{p} = (p_1, \ldots, p_n) \in [k+1]^n$, we write

$$p_i' := \#\{j : p_j \leq i\},$$

$$\mathrm{inv}(\mathbf{p}) := \#\{(i,j) : 1 \leq i < j \leq n : p_i > p_j\},$$

$$D(\mathbf{p}) := \{i : p_i > p_{i+1}\}.$$

Let $\mathbf{ps}_{q;\ell}(F_\alpha) := F_\alpha(1, q, q^2, \ldots, q^{\ell-1}, 0, 0, \ldots)$. Then applying $\sum_{\ell \geq 0} u_1^\ell \mathbf{ps}_{q_1;\ell+1}$ and the specialization $t_i = q_2^i u_2$ for all $i = 0, 1, \ldots, n$ to Theorem 1.3.2, we recover the following result of Garsia and Gessel [25, Theorem 2.2] on the generating function of multivariate distribution of five permutation statistics:

$$\frac{\sum_{w \in \mathfrak{S}_n} q_0^{\mathrm{inv}(w)} q_1^{\mathrm{maj}(w^{-1})} u_1^{\mathrm{des}(w^{-1})} q_2^{\mathrm{maj}(w)} u_2^{\mathrm{des}(w)}}{(u_1; q_1)_n (u_2; q_2)_n} = \sum_{\ell, k \geq 0} u_1^\ell u_2^k \sum_{(\lambda, \mu) \in B(\ell, k)} q_0^{\mathrm{inv}(\mu)} q_1^{|\lambda|} q_2^{|\mu|}.$$

Here $(u;q)_n := \prod_{0 \leq i \leq n} (1 - q^i u)$, the set $B(\ell, k)$ consists of all pairs of weak compositions $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\mu = (\mu_1, \ldots, \mu_n)$ satisfying the conditions $\ell \geq \lambda_1 \geq \cdots \geq \lambda_n$, $\max\{\mu_i : 1 \leq i \leq n\} \leq k$, and $\lambda_i = \lambda_{i+1} \Rightarrow \mu_i \geq \mu_{i+1}$ (such pairs (λ, μ) are sometimes called *bipartite partitions*), and $\mathrm{inv}(\mu) := \#\{(i,j) : 1 \leq i < j \leq n,\ \mu_i > \mu_j\}$. Some further specializations of Theorem 1.3.2 imply identities of MacMahon-Carlitz and Adin-Brenti-Roichman (see §4.2.6).

Finally let W be any finite Coxeter group with Coxeter complex $\Delta(W)$ and Hecke algebra $H_W(q)$. In Chapter 5, we generalize our $H_n(0)$-action on $\mathbb{F}[\mathcal{B}_n]$ to an $H_W(q)$-action on the Stanley-Reisner ring of $\Delta(W)$. We show that the invariant algebra of this $H_W(q)$-action is essentially the same as the invariant algebra of the $\mathfrak{S}_n$-action on $\mathbb{F}[\Delta(W)]$, and prove that the coinvariant algebra of this $H_W(q)$-action carries the regular representation of $H_W(q)$ for generic q. We also study the special case when $q = 0$.

Chapter 2

Preliminaries

2.1 Finite Coxeter groups

A *finite Coxeter group* is a finite group with the following presentation

$$W := \langle s_1, \ldots, s_d : s_i^2 = 1,\ (s_i s_j s_i \cdots)_{m_{ij}} = (s_j s_i s_j \cdots)_{m_{ij}}, 1 \leq i \neq j \leq d \rangle$$

where $m_{ij} \in \{2, 3, \ldots\}$ and $(aba\cdots)_m$ is an alternating product of m terms. We call (W, S) a *finite Coxeter system*, where $S := \{s_1, \ldots, s_d\}$. We often identify a subset J of S with the set $\{j : s_j \in J\}$ of the subscripts of the elements in J. If an expression $w = s_{i_1} \cdots s_{i_k}$ is the shortest one among all such expressions, then it is said to be *reduced* and k is the *length* $\ell(w)$ of w. The *descent set* of w is $D(w) := \{s \in S : \ell(ws) < \ell(w)\}$ and its elements are called the *descents* of w.

The *(left) weak order* is a partial order on W such that $u \leq w$ if and only if $\ell(w) = \ell(u) + \ell(wu^{-1})$. Given $J \subseteq S$, the *descent class* of J consists of the elements of W with descent set J, and turns out to be an interval under the weak order of W, denoted by $[w_0(J), w_1(J)]$. One sees that $w_0(J)$ is the longest element of the *parabolic subgroup* $W_J := \langle J \rangle$ of W. See Björner and Wachs [11, Theorem 6.2].

Every (left) W_J-coset has a unique *minimal coset representative*, i.e. a unique element of the smallest length. The minimal coset representatives for all W_J-cosets form the set $W^J := \{w \in W : D(w) \subseteq J^c\}$, where $J^c := S \setminus J$. Every element of W can be written uniquely as $w = w^J w_J$ such that $w^J \in W^J$ and $w_J \in W_J$, and one has $\ell(w) = \ell(w^J) + \ell(w_J)$. We need the following lemma for later use.

Lemma 2.1.1. *Let $J \subseteq I \subseteq S$, $s_i \in S$, and $w \in W^J$. If $i \notin D(w^{-1})$ and $s_iw \notin W^J$, then $s_iw = ws_j$ for some $s_j \in J$, which implies $s_iwW_I = wW_I$.*

Proof. If $i \notin D(w^{-1})$ and $s_iw \notin W_J$, then $s_iw = us_j$ for some $s_j \in J$ and $u \in W$ with $\ell(u) = \ell(w)$. Since $w \in W^J$, we have $u = s_iws_j = w$ by the deletion property of W [10, Proposition 1.4.7]. Hence $s_iwW_I = ws_jW_I = wW_I$. □

There is a well-known classification of the irreducible finite Coxeter groups. The *symmetric group* $\mathfrak{S}_n$ is the Coxeter group of type A_{n-1}. It consists of all permutations of the set $[n] := \{1, 2, \ldots, n\}$, and is generated by $S = \{s_1, \ldots, s_{n-1}\}$, where s_i is the adjacent transposition $(i, i+1)$. The generating set S satisfies the Coxeter presentation given in the beginning of this section, with $m_{ij} = 2$ if $|i-j| > 1$ and $m_{ij} = 3$ if $|i-j| = 1$.

If $w \in \mathfrak{S}_n$ then $\ell(w) = \mathrm{inv}(w) := \#\mathrm{Inv}(w)$, where

$$\mathrm{Inv}(w) := \{(i,j) : 1 \leq i < j \leq n,\ w(i) > w(j)\}$$

and the descent set of w is

$$D(w) = \{i : 1 \leq i \leq n-1,\ w(i) > w(i+1)\}.$$

In type A it is often convenient to use compositions. A *composition* is a sequence $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ of positive integers $\alpha_1, \ldots, \alpha_\ell$. The *length* of α is $\ell(\alpha) := \ell$ and the *size* of α is $|\alpha| := \alpha_1 + \cdots + \alpha_\ell$. If the size of α is n then we say α is a composition of n and write $\alpha \models n$.

Let $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ be a composition of n, and write $\sigma_j := \alpha_1 + \cdots + \alpha_j$ for $j = 0, 1, \ldots, \ell$; in particular, $\sigma_0 = 0$ and $\sigma_\ell = n$. Let $D(\alpha) := \{\sigma_1, \ldots, \sigma_{\ell-1}\}$ be the *descent set* of α. The map $\alpha \mapsto D(\alpha)$ is a bijection between compositions of n and subsets of $[n-1]$. Write $\alpha \preccurlyeq \beta$ if α and β are both compositions of n with $D(\alpha) \subseteq D(\beta)$.

The parabolic subgroup $\mathfrak{S}_\alpha := \langle s_i : i \in [n-1] \setminus D(\alpha) \rangle$ consists of all permutations $w \in \mathfrak{S}_n$ satisfying

$$\{w(\sigma_{j-1}+1), \ldots, w(\sigma_j)\} = \{\sigma_{j-1}+1, \ldots, \sigma_j\}, \quad j = 1, \ldots, \ell.$$

The set of minimal (left) $\mathfrak{S}_\alpha$-coset representatives is $\mathfrak{S}^\alpha := \{w \in \mathfrak{S}_n : D(w) \subseteq D(\alpha)\}$.

2.2 Stanley-Reisner rings

An *(abstract) simplicial complex* Δ on a vertex set V is a collection of finite subsets of V, called *faces*, such that a subset of a face is also a face. A maximal face is called a *chamber*. If $F \subseteq F' \in \Delta$, then $[F, F']$ consists of all faces between F and F'. The dimension of a face F is $|F| - 1$, and the dimension of Δ is the maximum dimension of its faces.

A simplicial complex is *pure d-dimensional* if every face is contained in a d-dimensional chamber. A pure $(d-1)$-dimensional complex is *balanced* if there exists a coloring map $r : V \to [d]$ such that every chamber consists of vertices of distinct colors. The *rank set* of a face F, denote by $r(F)$, is the set of all colors of its vertices. If J is a subset of $[d]$ then the *rank-selected subcomplex* Δ_J consists of all faces whose rank is contained in J.

Let Δ be a finite or countably infinite simplicial complex. An ordering of its chambers $C_1, C_2, \ldots$ is a *shelling order* if for all $k = 1, 2, \ldots$ we have that $\Delta_{k-1} \cap C_k$ is pure $(\dim C_k - 1)$-dimensional, where Δ_{k-1} is the subcomplex generated by $C_1, \ldots, C_{k-1}$. Given such a shelling order, define the *restriction* of a chamber C_k to be the face

$$R(C_k) := \{v \in C_k : C_k \setminus \{v\} \in \Delta_{k-1}\}. \tag{2.1}$$

Then the union $\cup_{i=1}^{k} [R(C_i), C_i]$ is disjoint and equal to Δ_k for all k. Conversely, any ordering $C_1, C_2, \ldots$ of the chambers satisfying this property must be a shelling.

Example 2.2.1. An example of balanced shellable complex is the *Coxeter complex* $\Delta(W)$ of a finite Coxeter system (W, S), where $S = \{s_1, \ldots, s_d\}$. Its faces are the parabolic cosets wW_J for all $w \in W$ and $J \subseteq S$; a face wW_J is contained in another face uW_I if $wW_J \supseteq uW_I$. The vertices of $\Delta(W)$ are the maximal proper parabolic cosets wW_{i^c} for all $w \in W$ and $i \in [d]$, where $i^c := S \setminus \{s_i\}$. Coloring the vertices by $r(wW_{i^c}) = i$ one sees that $\Delta(W)$ is balanced. A face wW_J has vertices wW_{i^c} for all $i \in J^c$, and has rank multiset $r(wW_J) = J^c$. The chambers of $\Delta(W)$ are the elements in W, and Björner [9, Theorem 2.1] showed that any linear extension of the weak order of W gives a shelling order of $\Delta(W)$.

Let Δ be a finite simplicial complex on the vertex V. The *Stanley-Reisner ring* of Δ over an arbitrary field $\mathbb{F}$ is defined as $\mathbb{F}[\Delta] := \mathbb{F}[V]/I_\Delta$, where

$$I_\Delta := \left(vv' : v, v' \in V, \{v, v'\} \notin \Delta \right).$$

It has an $\mathbb{F}$-basis of all nonzero monomials. A monomial $m = v_1 \cdots v_k$ in $\mathbb{F}[\Delta]$ is nonzero if and only if the vertices $v_1, \ldots, v_k$ belong to the same chamber, i.e. the set of $v_1, \ldots, v_k$ is a face of Δ; we denote this face by $\mathrm{supp}(m)$.

Now assume Δ is balanced with coloring map $r : V \to [d]$. A nonzero monomial $m = v_1 \cdots v_k$ has *rank multiset* $r(m)$ consisting of all the ranks $r(v_1), \ldots, r(v_k)$, and is multigraded by $t_{r(v_1)} \cdots t_{r(v_k)}$. This defines a multigrading on $\mathbb{F}[\Delta]$, with homogeneous components indexed by multisets of ranks. Such a monomial m is encoded by the pair $(\mathrm{supp}(m), r(m))$. The *rank polynomials* in $\mathbb{F}[\Delta]$ are the homogeneous elements

$$\theta_i := \sum_{r(v)=i} v, \quad i = 1, \ldots, d.$$

One sees that $\theta_1^{a_1} \cdots \theta_d^{a_d}$ equals the sum of all nonzero monomials m with $r(m) = \{1^{a_1}, \ldots, d^{a_d}\}$. Hence $\mathbb{F}[\Theta]$ is a polynomial subalgebra of $\mathbb{F}[\Delta]$, where $\Theta := \{\theta_1, \ldots, \theta_d\}$.

Theorem 2.2.2 (Garsia [24] and Kind-Kleinschmidt [38]). *If Δ is balanced and shellable, then the Stanley-Reisner ring $\mathbb{F}[\Delta]$ is a free $\mathbb{F}[\Theta]$-module with a basis*

$$\{y_{R(C)} : \ C \textit{ is a chamber of } \Delta\}$$

where $R(C)$ is the restriction of C from (2.1), and $y_{R(C)} := y_{v_1} \cdots y_{v_k}$ if $R(C)$ is a chamber with distinct vertices $v_1, \ldots, v_k$.

If Δ is balanced and shellable, then so is any rank-selected subcomplex Δ_J. For each chamber F of Δ_J, there exists a unique chamber $C(F)$ of Δ such that $R(C(F)) \subseteq F \subseteq C(F)$. A shelling order of Δ_J can be defined by saying $F \le F'$ whenever $C(F) \le C(F')$, and the restriction map is given by $R_J(F) = R(C(F))$. See Björner [9, Theorem 1.6]. Therefore Theorem 2.2.2 implies that $\mathbb{F}[\Delta_J]$ is a free module over $\mathbb{F}[\Theta_J] := \mathbb{F}[\theta_j : j \in J]$ with a basis of monomials indexed by $R(C(F))$ for all chambers F of Δ_J.

The Stanley-Reisner ring $\mathbb{F}[P]$ of a finite poset P is the same as the Stanley-Reisner ring of its *order complex*, whose faces are the chains in P ordered by reverse refinement. Explicitly,

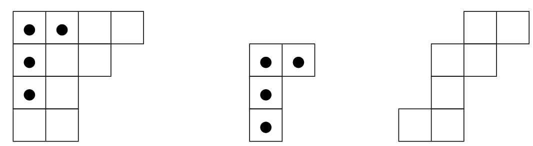
$$\mathbb{F}[P] := \mathbb{F}[y_v : v \in P]/(y_u y_v : \ u \text{ and } v \text{ are incomparable in } P).$$

The nonzero monomials in $\mathbb{F}[P]$ are indexed by multichains of P. Multiplying nonzero monomials corresponds to merging the indexing multichains; the result is zero if the multichains cannot be merged together. If P is ranked then its order complex is balanced and its Stanley-Reisner ring $\mathbb{F}[P]$ is multigraded.

2.3 Ribbon numbers

A *partition* of n is a set of positive integers, often written as a descreasing sequence $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, such that the *size* $|\lambda| := \lambda_1 + \cdots \lambda_\ell$ equals n; it is denoted by $\lambda \vdash n$.

A partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ can be represented by its *Young diagram*, which has λ_i boxes on its i-th row. If μ is a partition whose Young diagram is contained in the Young diagram of λ, then one has a skew shape λ/μ. For example,



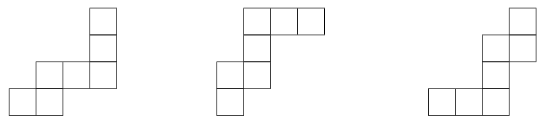
$\lambda = (4,3,2,2)$ $\quad$ $\mu = (2,1)$ $\quad$ λ/μ

A *semistandard Young tableau* τ of an arbitrary skew shape λ/μ is a filling of the skew diagram of λ/μ by positive integers such that every row weakly increases from left to right and every column strictly increases from top to bottom. Reading these integers from the bottom row to the top row and proceeding from left to right within each row gives the *reading word* $w(\tau)$ of τ. Say τ is a *standard Young tableau* if the integers appearing in τ are precisely $1, \ldots, n$ without repetition, i.e. $w(\tau) \in \mathfrak{S}_n$. The *descents* of a standard Young tableau τ are those numbers i appearing in a higher row of τ than $i+1$, or in other words, the descents of $w(\tau)^{-1}$. The *major index* $\mathrm{maj}(\tau)$ of a standard Young tableau τ is the sum of all its descents. Denote by $\mathrm{SSYT}(\lambda/\mu)$ [$\mathrm{SYT}(\lambda/\mu)$ resp.] the set of all semistandard [standard resp.] Young tableaux of shape λ/μ.

A *ribbon* is a skew connected diagram without 2×2 boxes. A ribbon α whose rows have lengths $\alpha_1, \ldots, \alpha_\ell$, ordered *from bottom to top*, is identified with the composition $\alpha = (\alpha_1, \ldots, \alpha_\ell)$. Denote by α^c the composition of n with $D(\alpha^c) = [n-1] \setminus D(\alpha)$, and write

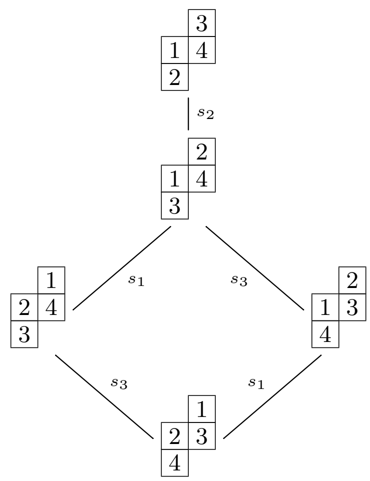
$$\overleftarrow{\alpha} := (\alpha_\ell, \ldots, \alpha_1), \qquad \alpha' := \overleftarrow{\alpha^c} = \overleftarrow{\alpha}^c.$$

One can check that the ribbon diagram of α' is the *transpose* of the ribbon α. An example is given below.



$\alpha = (2,3,1,1)$ $\quad$ $\alpha^c = (1,2,1,3)$ $\quad$ $\alpha' = (3,1,2,1)$

A *(standard) ribbon tableau* is a standard Young tableau of ribbon shape α. Taking the reading word $\tau \mapsto w(\tau)$ gives a bijection between $\mathrm{SYT}(\alpha)$ and the *descent class* of α, which consists of all permutations w in $\mathfrak{S}_n$ with $D(w) = D(\alpha)$. We know the descent class of α is an interval $[w_0(\alpha), w_1(\alpha)]$ under weak order of $\mathfrak{S}_n$. For instance, the descent class of $\alpha = (1,2,1)$ is given below.



In particular, the ribbon tableaux of $w_0(\alpha)$ and $w_1(\alpha)$ can be respectively obtained by

- filling with $1, 2, \ldots, n$ the columns of the ribbon α from top to bottom, starting with the leftmost column and proceeding toward the rightmost column,
- filling with $1, 2, \ldots, n$ the rows of the ribbon α from left to right, starting with the top row and proceeding toward the bottom row.

Now we recall from Reiner and Stanton [50, §9, §10] the definitions and properties of various ribbon numbers. Let $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ be a composition of n, and write $\sigma_i := \alpha_1 + \cdots + \alpha_i$ for $i = 0, \ldots, \ell$. The *ribbon number* r_α is the cardinality of the descent class of α. The *q-ribbon number* is

$$r_\alpha(q) := \sum_{\substack{w \in \mathfrak{S}_n: \\ D(w) = D(\alpha)}} q^{\mathrm{inv}(w)} = [n]!_q \det\left(\frac{1}{[\sigma_j - \sigma_{i-1}]!_q}\right)_{i,j=1}^{\ell} \tag{2.2}$$

where $[n]!_q = [n]_q[n-1]_q \cdots [1]_q$ and $[n]_q = 1+q+\cdots+q^{n-1}$. By Foata and Schützenberger [23], the two permutation statistics inv and maj are equidistributed on every inverse descent class $\{w \in \mathfrak{S}_n : D(w^{-1}) = D(\alpha)\}$. Thus

$$r_\alpha(t) = \sum_{\substack{w\in\mathfrak{S}_n:\\ D(w)=D(\alpha)}} t^{\,\mathrm{maj}(w^{-1})} \xlongequal{w(\tau)\leftrightarrow\tau} \sum_{\tau\in\mathrm{SYT}(\alpha)} t^{\mathrm{maj}(\tau)}.$$

A further generalization, introduced by Reiner and Stanton [50], is the (q,t)*-ribbon number*

$$r_\alpha(q,t) := \sum_{\substack{w\in\mathfrak{S}_n:\\ D(w)=D(\alpha)}} \mathrm{wt}(w;q,t) = n!_{q,t} \det\left(\varphi^{\sigma_{i-1}} \frac{1}{(\sigma_j - \sigma_{i-1})!_{q,t}}\right)_{i,j=1}^{\ell}.$$

Here $\mathrm{wt}(w;q,t)$ is some weight defined by a product expression, $\varphi : t \mapsto t^q$ is the *Frobenius operator*, and $m!_{q,t} := (1 - t^{q^m-1})(1 - t^{q^m-q}) \cdots (1 - t^{q^m-q^{m-1}})$.

All these ribbon numbers can be interpreted by the *homology representation* χ^α [χ^α_q resp.] of $\mathfrak{S}_n$ [$G = GL(n,\mathbb{F}_q)$ resp.], defined as the top homology of the *rank-selected Coxeter complex* $\Delta(\mathfrak{S}_n)_\alpha$ [*Tits building* $\Delta(G)_\alpha$ resp.], and by the intertwiner $M^\alpha = \mathrm{Hom}_{\mathbb{F}\mathfrak{S}_n}(\chi^\alpha, \mathbb{F}[X])$ [$M^\alpha_q = \mathrm{Hom}_{\mathbb{F}G}\left(\chi^\alpha_q, \mathbb{F}[X]\right)$ resp.] as a module over $\mathbb{F}[X]^{\mathfrak{S}_n}$ [$\mathbb{F}[X]^G$ resp.]. By work of Reiner and Stanton [50], one has the following picture.

$$\begin{tikzcd}
 & \boxed{r_\alpha = \dim\chi^\alpha} & \\
\boxed{\begin{array}{c} r_\alpha(q) = \\ \dim\chi^\alpha_q \end{array}} \arrow[ur, squiggly, "q\to 1"] \arrow[rr, leftrightarrow, "q\leftrightarrow t"] & & \boxed{\begin{array}{c} r_\alpha(t) = \\ \mathrm{Hilb}(M^\alpha/\mathbb{F}[X]_+^{\mathfrak{S}_n} M^\alpha, t) \end{array}} \arrow[ul, squiggly, "t\to 1"'] \\
 & \boxed{\begin{array}{c} r_\alpha(q,t) = \\ \mathrm{Hilb}(M^\alpha_q/\mathbb{F}[X]_+^G M^\alpha_q, t) \end{array}} \arrow[ul, squiggly, "t\to 1"] \arrow[ur, squiggly, "{t\to t^{\frac{1}{q-1}},\ q\to 1}"'] &
\end{tikzcd}$$

We will provide a similar interpretation of these ribbon numbers by representations of the 0-Hecke algebra $H_n(0)$.

The ribbon numbers are related to the multinomial coefficients by inclusion-exclusion. Let $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ be a composition of n. One has the *multinomial and q-multinomial coefficients*

$$\begin{bmatrix} n \\ \alpha \end{bmatrix} := \frac{n!}{\alpha_1! \cdots \alpha_\ell!} = \#\{w \in \mathfrak{S}_n : D(w) \subseteq D(\alpha)\},$$

$$\begin{bmatrix} n \\ \alpha \end{bmatrix}_q := \frac{[n]!_q}{[\alpha_1]!_q \cdots [\alpha_\ell]!_q} = \sum_{\substack{w\in\mathfrak{S}_n: \\ D(w)\subseteq D(\alpha)}} q^{\mathrm{inv}(w)}.$$

Reiner and Stanton [50] introduced the (q,t)-*multinomial coefficient*

$$\begin{bmatrix} n \\ \alpha \end{bmatrix}_{q,t} := \frac{n!_{q,t}}{\alpha_1!_{q,t} \cdot \varphi^{\sigma_1}(\alpha_2!_{q,t}) \cdots \varphi^{\sigma_{\ell-1}}(\alpha_\ell!_{q,t})} = \sum_{w\in\mathfrak{S}_n: D(w)\subseteq D(\alpha)} \mathrm{wt}(w;q,t).$$

Assume q is a primer power below. Let $G = GL(n,\mathbb{F}_q)$ be the finite general linear group over $\mathbb{F}_q$, and let P_α be the parabolic subgroup of all invertible block upper triangular matrices whose diagonal blocks have sizes given by the composition α. Then

$$\begin{bmatrix} n \\ \alpha \end{bmatrix}_q = |G/P_\alpha|,$$

$$\begin{bmatrix} n \\ \alpha \end{bmatrix}_t = \mathrm{Hilb}\left(\mathbb{F}[X]^{\mathfrak{S}_\alpha}/(\mathbb{F}[X]^{\mathfrak{S}_n}_+), t\right),$$

$$\begin{bmatrix} n \\ \alpha \end{bmatrix}_{q,t} = \mathrm{Hilb}\left(\mathbb{F}[X]^{P_\alpha}/(\mathbb{F}[X]^{G}_+), t\right).$$

2.4 Hecke algebra

Suppose that $\mathbb{F}$ is an arbitrary field and q is an indeterminate. Let

$$W := \langle S : s_i^2 = 1,\ (s_is_js_i\cdots)_{m_{ij}} = (s_js_is_j\cdots)_{m_{ij}}, 1\le i\ne j\le d\rangle$$

be a finite Coxeter group generated by $S := \{s_1,\dots,s_d\}$. The *Hecke algebra* $H_W(q)$ of W is the associative $\mathbb{F}(q)$-algebra generated by $T_1,\dots,T_d$ with relations

$$\begin{cases} (T_i+1)(T_i-q) = 0, & 1\le i\le d, \\ (T_iT_jT_i\cdots)_{m_{ij}} = (T_jT_iT_j\cdots)_{m_{ij}}, & 1\le i\ne j\le d. \end{cases}$$

By Tits' solution to the word problem for W [59], the element $T_w := T_{i_1}\cdots T_{i_k}$ is well defined for any $w\in W$ with a reduced expression $w = s_{i_1}\cdots s_{i_k}$. By Bourbaki [12, Exercise 23, p. 55], the set $\{T_w : w\in W\}$ is an $\mathbb{F}(q)$-basis for $H_W(q)$. See also Humphreys [36, Chapter 7]. The *group algebra* $\mathbb{F}W$ of W and the 0-*Hecke algebra* $H_W(0)$ are specializations of the Hecke algebra $H_W(q)$ at $q=1$ and $q=0$, respectively.

The *trivial representation of* $H_W(q)$ is the one dimensional $H_W(q)$-module on which T_i acts by q for all i; it specializes to the trivial representation of W when $q = 1$.

For every $J \subseteq S$, define $H_{W,J}(q)$ to be the *parabolic subalgebra* of $H_W(q)$ generated by $\{T_j : s_j \in J\}$. Let

$$\sigma_J := \sum_{w \in W_J} T_w. \tag{2.3}$$

Then

$$T_w \sigma_J = q^{\ell(w)} \sigma_J, \quad \forall w \in W_J.$$

Thus $H_{W,J}(q)\sigma_J$ is the trivial representation of $H_{W,J}(q)$. The induction of $H_{W,J}(q)\sigma_J$ to $H_W(q)$ gives the *parabolic representation* $H_W(q)\sigma_J$, which is a left ideal of $H_W(q)$ with $\mathbb{F}(q)$-basis $\{T_w\sigma_J : w \in W^J\}$.

Using these parabolic representations Mathas [47] defined a chain complex (Ω_*, ∂_*) which is a q-analogue of (the chain complex of) the Coxeter complex $\Delta(W)$. It has a rank-selected subcomplex $\Omega_*(J)$ for every $J \subseteq S$, whose homology is vanishing everywhere except in the top degree $|J^c|$. The top homology Π_J of $\Omega_*(J)$ is an $H_W(q)$-submodule of $H_W(q)\sigma_J$ with an $\mathbb{F}(q)$-basis

$$\left\{ \xi_u = \sum_{w \in W_{J^c}} (-q)^{\ell(w)} T_{uw}\sigma_J : u \in W,\ D(u) = J^c \right\}. \tag{2.4}$$

Mathas [47] showed that the decomposition of $H_W(q)$-modules

$$H_W(q) = \bigoplus_{J \subseteq S} \Pi_J \tag{2.5}$$

holds in the following cases:

- if $q = 0$ then this is precisely Norton's decomposition (1.1) of the 0-Hecke algebra $H_W(0)$ (see also §2.5);
- if $\mathbb{F} = \mathbb{Q}$ and $q = 1$ then it is the decomposition of the group algebra $\mathbb{Q}W$ by Solomon [53];
- it also holds for all semisimple specializations of $H_W(q)$.

We will consider $H_W(q)$-actions on multigraded $\mathbb{F}(q)$-vector spaces whose homogeneous components are all $H_W(q)$-stable; such a vector space is called a *multigraded* $H_W(q)$*-module*.

2.5 0-Hecke algebras

Let $\mathbb{F}$ be an arbitrary field and let

$$W := \langle S : s_i^2 = 1,\ (s_is_js_i\cdots)_{m_{ij}} = (s_js_is_j\cdots)_{m_{ij}}, 1 \le i \ne j \le d\rangle$$

be a finite Coxeter group generated by $S := \{s_1, \ldots, s_d\}$. The 0-*Hecke algebra* $H_W(0)$ of W is the specialization of the Hecke algebra $H_W(q)$ at $q = 0$, namely the associative $\mathbb{F}$-algebra generated by $\overline{\pi}_1, \ldots, \overline{\pi}_d$ with relations

$$\begin{cases} \overline{\pi}_i^2 = -\overline{\pi}, & 1 \le i \le d, \\ (\overline{\pi}_i\overline{\pi}_j\overline{\pi}_i\cdots)_{m_{ij}} = (\overline{\pi}_j\overline{\pi}_i\overline{\pi}_j\cdots)_{m_{ij}}, & 1 \le i \ne j \le d. \end{cases}$$

Here $\overline{\pi}_i = T_i|_{q=0}$. Let $\pi_i := \overline{\pi}_i + 1$. Then $\pi_1, \ldots, \pi_d$ form another generating set for $H_W(0)$, with relations

$$\begin{cases} \pi_i^2 = \pi, & 1 \le i \le d, \\ (\pi_i\pi_j\pi_i\cdots)_{m_{ij}} = (\pi_j\pi_i\pi_j\cdots)_{m_{ij}}, & 1 \le i \ne j \le d. \end{cases}$$

If an element $w \in W$ has a reduced expression $w = s_{i_1}\cdots s_{i_k}$ then $\overline{\pi}_w := \overline{\pi}_{i_1}\cdots\overline{\pi}_{i_k}$ and $\pi_w := \pi_{i_1}\cdots\pi_{i_k}$ are well defined. Both sets $\{\overline{\pi}_w : w \in W\}$ and $\{\pi_w : w \in W\}$ are $\mathbb{F}$-bases for $H_W(0)$. One can check that π_w equals the sum of $\overline{\pi}_u$ over all u less than or equal to w in the Bruhat order of W. In particular, for all $J \subseteq S$,

$$\pi_{w_0(J)} = \sum_{u \in W_J} \overline{\pi}_u$$

is the specialization at $q = 0$ of the element σ_J in $H_W(q)$ defined in (2.3).

Norton [49] decomposed the 0-Hecke algebra $H_W(0)$ into a direct sum of pairwise non-isomorphic indecomposable (left) $H_W(0)$-submodules

$$H_W(0) = \bigoplus_{J \subseteq S} \mathbf{P}_J.$$

Each direct summand $\mathbf{P}_J := H_W(0) \cdot \overline{\pi}_{w_0(J)}\pi_{w_0(J^c)}$ has an $\mathbb{F}$-basis

$$\left\{\overline{\pi}_w\pi_{w_0(J^c)} : w \in [w_0(J), w_1(J)]\right\}.$$

Its *radical* $\operatorname{rad}\mathbf{P}_J$ is defined as the intersection of all maximal $H_n(0)$-submodules in general, and turns out to be the unique maximal $H_n(0)$-submodule spanned by

$$\{\overline{\pi}_w\pi_{w_0(J)} : w \in (w_0(J), w_1(J)]\}$$

in this case. Although $\mathbf{P}_J$ itself is not necessarily simple, its *top* $\mathbf{C}_J := \mathbf{P}_J / \operatorname{rad} \mathbf{P}_J$ is a one-dimensional simple $H_n(0)$-module with the action of $H_n(0)$ given by

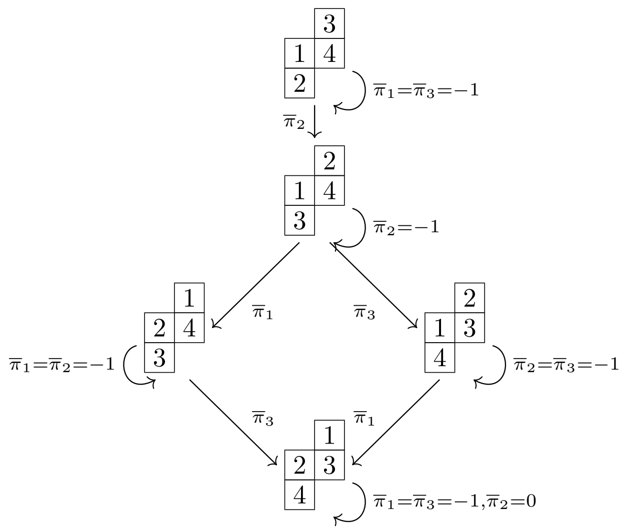
$$\overline{\pi}_i = \begin{cases} -1, & \text{if } i \in J, \\ 0, & \text{if } i \notin J. \end{cases}$$

It follows from general representation theory of associative algebras (see e.g. [5, §I.5]) that $\{\mathbf{P}_J : J \subseteq S\}$ is a complete list of non-isomorphic projective indecomposable $H_W(0)$-modules and $\{\mathbf{C}_J : J \subseteq S\}$ is a complete list of non-isomorphic simple $H_W(0)$-modules.

The Hecke algebra of the symmetric group $\mathfrak{S}_n$ is denoted by $H_n(q)$ and the 0-Hecke algebra of $\mathfrak{S}_n$ is denoted by $H_n(0)$. For every composition α of n, we denote by $\mathbf{C}_\alpha$ and $\mathbf{P}_\alpha$ the simple and projective indecomposable $H_n(0)$-modules indexed by $D(\alpha) \subseteq [n-1]$. One can realize $\mathbf{P}_\alpha$ in a combinatorial way [31, 39] using ribbon tableaux. We know that the ribbon tableaux of shape α are in bijection with the descent class of α, hence in bijection with the basis $\{\pi_w \overline{\pi}_{w_0(\alpha^c)} : D(w) = D(\alpha)\}$ of $\mathbf{P}_\alpha$. The $H_n(0)$-action on $\mathbf{P}_\alpha$ agrees with the following $H_n(0)$-action on these ribbon tableaux:

$$\overline{\pi}_i \tau = \begin{cases} -\tau, & \text{if } i \text{ is in a higher row of } \tau \text{ than } i+1, \\ 0, & \text{if } i \text{ is in the same row of } \tau \text{ as } i+1, \\ s_i \tau, & \text{if } i \text{ is in a lower row of } \tau \text{ than } i+1, \end{cases} \tag{2.6}$$

where τ is a ribbon tableau of shape α and $s_i\tau$ is obtained from τ by swapping i and $i+1$. This action gives rise to a directed version of the Hasse diagram of the interval $[w_0(\alpha), w_1(\alpha)]$ under the weak order. The top tableau in this diagram corresponds to $\mathbf{C}_\alpha = \operatorname{top}(\mathbf{P}_\alpha)$, and the bottom tableau spans the *socle* $\operatorname{soc}(\mathbf{P}_\alpha) \cong \mathbf{C}_{\overleftarrow{\alpha}}$ of $\mathbf{P}_\alpha$, which is the unique minimal submodule of $\mathbf{P}_\alpha$. An example is given below for $\alpha = (1, 2, 1)$.



2.6 Quasisymmetric functions and noncommutative symmetric functions

Let $\mathbb{Z}[[X]]$ be the ring of formal power series over $\mathbb{Z}$ in commutative variables $x_1, x_2, \ldots$. The Hopf algebra QSym of quasisymmetric functions is a free $\mathbb{Z}$-module with basis given by the *monomial quasisymmetric functions*

$$M_\alpha := \sum_{i_1 > \cdots > i_\ell \geq 1} x_{i_1}^{\alpha_1} \cdots x_{i_\ell}^{\alpha_\ell}$$

for all compositions $\alpha = (\alpha_1, \ldots, \alpha_\ell)$. By definition there is an inclusion $\mathrm{QSym} \subset \mathbb{Z}[[X]]$ of algebras. Another free $\mathbb{Z}$-basis consists of the *fundamental quasisymmetric functions*

$$F_\alpha := \sum_{\alpha \preccurlyeq \beta} M_\beta = \sum_{\substack{i_1 \geq \cdots \geq i_n \geq 1 \\ j \in D(\alpha) \Rightarrow i_j > i_{j+1}}} x_{i_1} \cdots x_{i_n}$$

for all compositions α, where $n = |\alpha|$, and $\alpha \preccurlyeq \beta$ means that α and β are both compositions of n with $D(\alpha) \subseteq D(\beta)$, or in other words, α is refined by β. Since $\alpha \mapsto D(\alpha)$ is a bijection, we sometimes write $F_I := F_\alpha$ if $I = D(\alpha) \subseteq [n-1]$ and n is clear from the context.

The reader might notice that the above definition for M_α and F_α is slightly different from the standard one, as the inequalities of the subscripts are reversed. This difference is certainly not essential, and the definition given here has the advantage that the principal specialization $F_\alpha(1,x,x^2,\ldots)$ involves $\mathrm{maj}(\alpha) := \sum_{i\in D(\alpha)} i$ rather than $\mathrm{maj}(\overleftarrow{\alpha})$, where $\overleftarrow{\alpha} := (\alpha_\ell,\ldots,\alpha_1)$ if $\alpha = (\alpha_1,\ldots,\alpha_\ell)$. We will use this in §4.2.6.

The product of QSym is the usual product of formal power series in commutative variables $x_1, x_2, \ldots$, and the coproduct for QSym is defined as $\Delta f(X) := f(X+Y)$ for all $f \in \mathrm{QSym}$, where $X + Y := \{x_0, x_1, \ldots, y_0, y_1, \ldots\}$ is a set of totally ordered commutative variables.

Let $\mathbb{Z}\langle\mathbf{X}\rangle$ be the free associative $\mathbb{Z}$-algebra in noncommutative variables $\mathbf{x}_1, \mathbf{x}_2, \ldots$. The Hopf algebra **NSym** is a subalgebra of $\mathbb{Z}\langle\mathbf{X}\rangle$, defined as the free associative algebra $\mathbb{Z}\langle\mathbf{h}_1, \mathbf{h}_2, \ldots\rangle$ where

$$\mathbf{h}_k := \sum_{1\le i_1\le\cdots\le i_k} \mathbf{x}_{i_1}\cdots\mathbf{x}_{i_k}.$$

It has a $\mathbb{Z}$-basis of the *complete homogeneous noncommutative symmetric functions* $\mathbf{h}_\alpha := \mathbf{h}_{\alpha_1}\cdots\mathbf{h}_{\alpha_\ell}$ for all compositions $\alpha = (\alpha_1,\ldots,\alpha_\ell)$. Another free $\mathbb{Z}$-basis consists of the *noncommutative ribbon Schur functions*

$$\mathbf{s}_\alpha := \sum_{\beta\preccurlyeq\alpha} (-1)^{\ell(\alpha)-\ell(\beta)}\mathbf{h}_\beta$$

for all compositions α. If $\alpha = (\alpha_1,\ldots,\alpha_\ell)$ and $\beta = (\beta_1,\ldots,\beta_k)$ are two compositions then

$$\mathbf{s}_\alpha\mathbf{s}_\beta = \mathbf{s}_{\alpha\beta} + \mathbf{s}_{\alpha\rhd\beta} \tag{2.7}$$

where

$$\alpha\beta := (\alpha_1,\ldots,\alpha_\ell,\beta_1,\ldots,\beta_k),$$

$$\alpha\rhd\beta := (\alpha_1,\ldots,\alpha_{\ell-1},\alpha_\ell+\beta_1,\beta_2,\ldots,\beta_k).$$

The coproduct of **NSym** is defined by $\Delta\mathbf{h}_k = \sum_{i=0}^k \mathbf{h}_i\otimes\mathbf{h}_{k-i}$, where $\mathbf{h}_0 := 1$.

The two Hopf algebras QSym and **NSym** are dual to each other via the pairing

$$\langle M_\alpha, \mathbf{h}_\beta\rangle = \langle F_\alpha, \mathbf{s}_\beta\rangle := \delta_{\alpha\beta}, \quad \forall\alpha,\ \beta.$$

They are related to the self-dual Hopf algebra Sym, the *ring of symmetric functions*, which is free $\mathbb{Z}$-module with a basis of *Schur functions* s_λ for all partitions λ of n. The product and coproduct of Sym are determined by

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda,$$

$$\Delta(s_\lambda) = \sum_{\mu,\nu} c^\lambda_{\mu\nu} s_\mu \otimes s_\nu,$$

where the coefficients $c^\lambda_{\mu\nu}$ are positive integers called the *Littlewood-Richardson coefficients.*

The definition of the Schur function s_λ is a special case of the *skew Schur function*

$$s_{\lambda/\mu} := \sum_{\tau \in \mathrm{SSYT}(\lambda/\mu)} x^\tau$$

of a skew shape λ/μ, where $x^\tau := x_1^{d_1} x_2^{d_2} \cdots$ if $d_1, d_2, \ldots$ have the multiplicities of $1, 2, \ldots$ in τ. One can take commutative images of elements in $\mathbb{Z}\langle \mathbf{X} \rangle$ by sending $\mathbf{x}_i$ to x_i for all i; this defines the *forgetful map* $\mathbb{Z}\langle X \rangle \to \mathbb{Z}[[X]]$. The commutative image of a noncommutative ribbon Schur function $\mathbf{s}_\alpha$ is nothing but the ribbon Schur function s_α. This gives a surjection $\mathbf{NSym} \twoheadrightarrow \mathrm{Sym}$ of Hopf algebras.

There is also a free $\mathbb{Z}$-basis for Sym consisting of the *monomial symmetric functions*

$$m_\lambda := \sum_{\lambda(\alpha)=\lambda} M_\alpha$$

for all partitions λ. Here $\lambda(\alpha)$ is the unique partition obtained from the composition α by rearranging its parts. This gives an inclusion $\mathrm{Sym} \hookrightarrow \mathrm{QSym}$ of Hopf algebras.

The relations between *Sym*, QSym, and **NSym** are summarized below.

$$\begin{tikzcd}
\mathbb{Z}\langle \mathbf{X} \rangle \arrow[r, "\chi"] & \mathbb{Z}[[X]] \\
\mathbf{NSym} \arrow[u, hook] \arrow[r, leftrightarrow, dashed, "\text{dual}"] \arrow[dr, two heads, "\chi"'] & \mathrm{QSym} \arrow[u, hook] \\
 & \mathrm{Sym} \arrow[u, hook]
\end{tikzcd}$$

We will use the following expansion of a Schur function indexed by a partition $\lambda \vdash n$:

$$s_\lambda = \sum_{\mu \vdash n} K_{\lambda\mu} m_\mu. \tag{2.8}$$

Here $K_{\lambda\mu}$ is the *Kostka number* which counts all semistandard Young tableaux of shape λ and type μ.

2.7 Characteristic maps

Now we recall the classic correspondence between (complex) $\mathfrak{S}_n$-representations and symmetric functions, and a similar correspondence by Krob and Thibon [39] for $H_n(0)$-representations (over any field $\mathbb{F}$). See also Bergeron and Li [7].

Let A be an $\mathbb{F}$-algebra and let $\mathcal{C}$ be a category of some finitely generated A-modules. The *Grothendieck group of* $\mathcal{C}$ is defined as the abelian group F/R, where F is the free abelian group on the isomorphism classes $[M]$ of the A-modules M in $\mathcal{C}$, and R is the subgroup of F generated by the elements $[M]-[L]-[N]$ corresponding to all exact sequences $0 \to L \to M \to N \to 0$ of A-modules in $\mathcal{C}$. Note that if A is semisimple, or if L, M, N are all projective A-modules, then $0 \to L \to M \to N \to 0$ implies $M \cong L \oplus N$. We often identify an A-module in $\mathcal{C}$ with the corresponding element in the Grothendieck group of $\mathcal{C}$.

Denote by $G_0(\mathfrak{S}_n)$ the Grothendieck group of the category of all finitely generated $\mathbb{C}\mathfrak{S}_n$-modules. The simple $\mathbb{C}\mathfrak{S}_n$-modules S_λ are indexed by partitions λ of n, and every finitely generated $\mathbb{C}\mathfrak{S}_n$-module is a direct sum of simple $\mathfrak{S}_n$-modules. Thus $G_0(\mathfrak{S}_n)$ is a free abelian group on the isomorphism classes $[S_\lambda]$ for all partitions λ of n. The tower of groups $\mathfrak{S}_\bullet : \mathfrak{S}_0 \hookrightarrow \mathfrak{S}_1 \hookrightarrow \mathfrak{S}_2 \hookrightarrow \cdots$ has a Grothendieck group

$$G_0(\mathfrak{S}_\bullet) := \bigoplus_{n\geq 0} G_0(\mathfrak{S}_n)$$

Using the natural inclusion $\mathfrak{S}_m \times \mathfrak{S}_n \hookrightarrow \mathfrak{S}_{m+n}$, one defines the product of S_μ and S_ν as the induction of $S_\mu \otimes S_\nu$ from $\mathfrak{S}_m \times \mathfrak{S}_n$ to $\mathfrak{S}_{m+n}$ for all partitions $\mu \vdash m$ and $\nu \vdash n$, and the coproduct of S_λ as the sum of its restriction to $\mathfrak{S}_i \times \mathfrak{S}_{n-i}$ for $i = 0, 1, \ldots, n$, for all partitions $\lambda \vdash n$. This gives $G_0(\mathfrak{S}_\bullet)$ a self-dual Hopf algebra structure.

The *Frobenius characteristic map* ch is defined by sending a simple S_λ to the Schur function s_λ, giving a Hopf algebra isomorphism between the Grothendieck group $G_0(\mathfrak{S}_\bullet)$ and the Hopf algebra Sym of symmetric functions.

The Grothendieck group of the category of all finitely generated $H_n(0)$-modules is denoted by $G_0(H_n(0))$, and the Grothendieck group of the category of finitely generated projective $H_n(0)$-modules is denoted by $K_0(H_n(0))$. By the result of Norton [49], one has

$$G_0(H_n(0)) = \bigoplus_{\alpha \models n} \mathbb{Z} \cdot [\mathbf{C}_\alpha], \quad K_0(H_n(0)) = \bigoplus_{\alpha \models n} \mathbb{Z} \cdot [\mathbf{P}_\alpha].$$

The tower of algebras $H_\bullet(0) : H_0(0) \hookrightarrow H_1(0) \hookrightarrow H_2(0) \hookrightarrow \cdots$ has two Grothendieck groups

$$G_0(H_\bullet(0)) := \bigoplus_{n\geq 0} G_0(H_n(0)), \quad K_0(H_\bullet(0)) := \bigoplus_{n\geq 0} K_0(H_n(0)).$$

These two Grothendieck groups are dual Hopf algebras with product and coproduct again given by induction and restriction of representations along the natural inclusions $H_m(0)\otimes H_n(0) \hookrightarrow H_{m+n}(0)$ of algebras. Krob and Thibon [39] introduced Hopf algebra isomorphisms

$$\mathrm{Ch} : G_0(H_\bullet(0)) \cong \mathrm{QSym}, \quad \mathbf{ch} : K_0(H_\bullet(0)) \cong \mathbf{NSym}$$

which we review next.

Let $M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_k \supseteq M_{k+1} = 0$ be a composition series of $H_n(0)$-modules with simple factors $M_i/M_{i+1} \cong \mathbf{C}_{\alpha^{(i)}}$ for $i = 0, 1, \ldots, k$. Then the *quasisymmetric characteristic* of M is

$$\mathrm{Ch}(M) := F_{\alpha^{(0)}} + \cdots + F_{\alpha^{(k)}}.$$

This is well defined by the Jordan-Hölder theorem. The *noncommutative characteristic* of a projective $H_n(0)$-module $M \cong \mathbf{P}_{\alpha^{(1)}} \oplus \cdots \oplus \mathbf{P}_{\alpha^{(k)}}$ is

$$\mathbf{ch}(M) := \mathbf{s}_{\alpha^{(1)}} + \cdots + \mathbf{s}_{\alpha^{(k)}}.$$

Krob and Thibon [39] also showed that $\mathrm{Ch}(\mathbf{P}_\alpha) = s_\alpha$ is the ribbon Schur function, which is the commutative image of $\mathbf{ch}(P_\alpha) = \mathbf{s}_\alpha$. Thus $\mathrm{Ch}(M)$ is symmetric if M is a finitely generated projective $H_n(0)$-module, but not vice versa: for instance, $\mathbf{C}_{12} \oplus \mathbf{C}_{21}$ is nonprojective but $\mathrm{Ch}(\mathbf{C}_{12} \oplus \mathbf{C}_{21}) = s_{21} \in \mathrm{Sym}$.

2.8 Cyclic $H_n(0)$-modules and the q-characteristic map

Let $H_n(0)^{(\ell)}$ be the $\mathbb{F}$-span of $\{\overline{\pi}_w : \ell(w) \geq \ell\}$. For a cyclic $H_n(0)$-module $N = H_n(0)v$, there is a length filtration $N^{(0)} \supseteq N^{(1)} \supseteq \cdots$ of $H_n(0)$-modules, where $N^{(\ell)} := H_n(0)^{(\ell)}v$. Krob and Thibon [39] defined the *length-graded quasisymmetric characteristic* of N as

$$\mathrm{Ch}_q(N) := \sum_{\ell\geq 0} q^\ell \mathrm{Ch}(N^{(\ell)}/N^{(\ell+1)}).$$

For example (c.f. Lemma 4.2.13), if α is a composition of n, then the cyclic $H_n(0)$-module $H_n(0)\pi_{w_0(\alpha^c)}$ has length-graded quasisymmetric characteristic

$$\mathrm{Ch}_q(H_n(0)\pi_{w_0(\alpha^c)}) = \sum_{w\in\mathfrak{S}^\alpha} q^{\mathrm{inv}(w)} F_{D(w^{-1})}.$$

In particular, taking $\alpha = 1^n$ one obtains

$$\begin{aligned}\mathrm{Ch}_q(H_n(0)) &= \sum_{w\in\mathfrak{S}_n} q^{\mathrm{inv}(w)} F_{D(w^{-1})} \\ &= \sum_{\alpha\models n} r_\alpha(q) F_\alpha.\end{aligned}$$

and then setting $q = 1$ one has

$$\mathrm{Ch}(H_n(0)) = \sum_{\alpha\models n} r_\alpha F_\alpha.$$

We often consider (multi)graded $H_n(0)$-modules M with countably many homogeneous components that are all finite dimensional. Since each component of M has a well-defined quasisymmetric characteristic and a (multi)grading, we obtain a (multi)graded quasisymmetric characteristic of M, which can be combined with the aforementioned length-graded quasisymmetric characteristic if in addition every homogeneous component is cyclic. If M is projective then one has a (multi)graded noncommutative characteristic of M. The (multi)graded Frobenius characteristic is defined in the same way for (multi)graded $\mathbb{C}\mathfrak{S}_n$-modules.

Chapter 3

0-Hecke algebra actions on coinvariants and flags

In this chapter we give interpretations of various ribbon numbers by studying 0-Hecke algebra actions on coinvariants and flags.

3.1 Coinvariant algebra of $H_n(0)$

The symmetric group $\mathfrak{S}_n$ acts on the polynomial ring $\mathbb{F}[X] := \mathbb{F}[x_1, \ldots, x_n]$ over an arbitrary field $\mathbb{F}$ by permuting the variables $x_1, \ldots, x_n$, and hence acts on the coinvariant algebra $\mathbb{F}[X]/(\mathbb{F}[X]_+^{\mathfrak{S}_n})$ of $\mathfrak{S}_n$, where $(\mathbb{F}[X]_+^{\mathfrak{S}_n})$ is the ideal generated by symmetric polynomials of positive degree. We often identify the polynomials in $\mathbb{F}[X]$ with their images in the coinvariant algebra of $\mathfrak{S}_n$ in this section.

The $H_n(0)$-action on the polynomial ring $\mathbb{F}[X]$ is via the *Demazure operators*

$$\pi_i(f) := \frac{x_i f - x_{i+1} s_i(f)}{x_i - x_{i+1}}, \quad \forall f \in \mathbb{F}[X],\ 1 \leq i \leq n-1.$$

It follows from this definition that

- $\deg(\pi_i f) = \deg(f)$ for all homogeneous polynomials $f \in \mathbb{F}[X]$,
- $\pi_i f = f$ if and only if $s_i f = f$ for all $f \in \mathbb{F}[X]$,
- $\pi_i(fg) = f\pi_i(g)$ for all $f, g \in \mathbb{F}[X]$ satisfying $\pi_i f = f$.

Thus this $H_n(0)$-action preserves the grading of $\mathbb{F}[X]$, has the *invariant algebra*

$$\mathbb{F}[X]^{H_n(0)} := \{f \in \mathbb{F}[X] : \pi_i f = f,\ i = 1, \ldots, n-1\} = \mathbb{F}[X]^{\mathfrak{S}_n}$$

equal to the coinvariant algebra of $\mathfrak{S}_n$, and is $\mathbb{F}[X]^{\mathfrak{S}_n}$-linear. Then one has a grading-preserving $H_n(0)$-action on the *coinvariant algebra of* $H_n(0)$, which is defined as

$$\mathbb{F}[X]/(f \in \mathbb{F}[X] : \deg f > 0,\ \pi_i f = f,\ 1 \leq i \leq n-1)$$

and coincides with the coinvariant algebra $\mathbb{F}[X]/(\mathbb{F}[X]_+^{\mathfrak{S}_n})$ of $\mathfrak{S}_n$.

To study this $H_n(0)$-action on its coinvariant algebra, we consider certain *Demazure atoms* which behave nicely under the $H_n(0)$-action. Specifically, we consider the polynomials

$$\overline{\pi}_w x_{D(w)}, \qquad \forall w \in \mathfrak{S}_n. \tag{3.1}$$

Here

$$x_I := \prod_{i \in I} x_1 \cdots x_i$$

for any $I \subseteq [n-1]$. See Mason [46] for more information on the Demazure atoms.

We will see in Lemma 3.1.1 that the Demazure atoms mentioned above are closely related to the *descent monomials*

$$w x_{D(w)} = \prod_{i \in D(w)} x_{w(1)} \cdots x_{w(i)}, \quad \forall w \in \mathfrak{S}_n.$$

It is well-known that the descent monomials form a basis for the coinvariant algebra of $\mathfrak{S}_n$; see e.g. Garsia [24] and Steinberg [57]. Allen [3] provided an elementary proof for this result, which we will adapt to the Demazure atoms $\overline{\pi}_w x_{D(w)}$. Thus we first recall Allen's proof below.

Recall from §1.1 that a *weak composition* is a finite sequence of nonnegative integers. A *partition* here is a finite decreasing sequence of nonnegative integers, with zeros ignored sometimes. Every monomial in $\mathbb{F}[X]$ can be written as $x^d = x_1^{d_1} \cdots x_n^{d_n}$ where $d = (d_1, \ldots, d_n)$ is a weak composition. Denote by $\lambda(d)$ the unique partition obtained from rearranging the weak composition d. Given two monomials x^d and x^e, write $x^d \prec x^e$ or $d \prec e$ if $\lambda(d) <_L \lambda(e)$, and write $x^d <_{ts} x^e$ if (i) $\lambda(d) <_L \lambda(e)$ or (ii) $\lambda(d) = \lambda(e)$ and $d <_L e$, where "$<_L$" is the lexicographic order.

Given a weak composition $d = (d_1, \ldots, d_n)$, we have a permutation $\sigma(d) \in \mathfrak{S}_n$ obtained by labelling $d_1, \ldots, d_n$ from the largest to the smallest, breaking ties from left to right. Construct a weak composition $\gamma(d)$ from this labelling as follows. First replace the largest label with 0, and recursively, if the label t has been replaced with s, then replace $t-1$ with s if it is to the left of t, or with $s+1$ otherwise. Let $\mu(d) = d - \gamma(d)$ be the component-wise difference. For example,

$$d = (3,1,3,0,2,0), \quad \sigma(d) = (1,4,2,5,3,6),$$

$$\gamma(d) = (1,0,1,0,1,0), \quad \mu(d) = (2,1,2,0,1,0).$$

The decomposition $d = \gamma(d) + \mu(d)$ is the usual *P-partition encoding* of d (see e.g. Stanley [55]), and $x^{\gamma(d)}$ is the descent monomial of $\sigma(d)^{-1}$. E.E. Allen [3, Proposition 2.1] showed that $w\mu(d) + \gamma(d) <_{ts} d$ for all $w \in \mathfrak{S}_n$ unless $w = 1$, and thus

$$m_{\mu(d)} \cdot x^{\gamma(d)} = x^d + \sum_{x^e <_{ts} x^d} c_e x^e, \quad c_e \in \mathbb{Z}, \tag{3.2}$$

where $m_{\mu(d)}$ is the monomial symmetric function corresponding to $\mu(d)$, i.e. the sum of the monomials in the $\mathfrak{S}_n$-orbit of $x^{\mu(d)}$. It follows that

$$\left\{ m_\mu \cdot w x_{D(w)} : \mu = (\mu_1 \geq \cdots \geq \mu_n \geq 0),\ w \in \mathfrak{S}_n \right\}$$

is triangularly related to the set of all monomials x^d, and thus an $\mathbb{F}$-basis for $\mathbb{F}[X]$. Therefore the descent monomials form an $\mathbb{F}[X]^{\mathfrak{S}_n}$-basis for $\mathbb{F}[X]$ and give an $\mathbb{F}$-basis for $\mathbb{F}[X]/(\mathbb{F}[X]^{\mathfrak{S}_n}_+)$.

Now we investigate the relation between our Demazure atoms and the descent monomials. First observe that if m is a monomial not containing x_i and x_{i+1}, then

$$\overline{\pi}_i(m x_i^a x_{i+1}^b) = \begin{cases} m(x_i^{a-1} x_{i+1}^{b+1} + x_i^{a-2} x_{i+1}^{b+2} \cdots + x_i^b x_{i+1}^a), & \text{if } a > b, \\ 0, & \text{if } a = b, \\ -m(x_i^a x_{i+1}^b - x_i^{a+1} x_{i+1}^{b-1} - \cdots - x_i^{b-1} x_{i+1}^{a+1}), & \text{if } a < b. \end{cases} \tag{3.3}$$

Lemma 3.1.1. *Suppose that α is a composition of n and w is a permutation in $\mathfrak{S}_n$ with $D(w) \subseteq D(\alpha)$. Then*

$$\overline{\pi}_w x_{D(\alpha)} = w x_{D(\alpha)} + \sum_{x^d \prec x_{D(\alpha)}} c_d x^d, \quad c_d \in \mathbb{Z}.$$

Moreover, $w x_{D(\alpha)}$ is a descent monomial if and only if $D(w) = D(\alpha)$.

Example 3.1.2. Let us look at some examples before proving this lemma.

First let $n=3$. If $\alpha=3$ then $x_{D(\alpha)}=1$. If $\alpha=12$ then

$$x_{D(\alpha)} = x_1 \xrightarrow{\overline{\pi}_1} x_2 \xrightarrow{\overline{\pi}_2} x_3.$$

If $\alpha=21$ then

$$x_{D(\alpha)} = x_1x_2 \xrightarrow{\overline{\pi}_2} x_1x_3 \xrightarrow{\overline{\pi}_1} x_2x_3.$$

If $\alpha=111$ then

$$\begin{tikzcd}
 & \underline{x_1x_2^2} \arrow[r, "\overline{\pi}_2"] & \underline{x_1x_3^2}+x_1x_2x_3 \arrow[dr, "\overline{\pi}_1"] & \\
x_{D(\alpha)} = \underline{x_1^2x_2} \arrow[ur, "\overline{\pi}_1"] \arrow[dr, "\overline{\pi}_2"'] & & & \underline{x_2x_3^2} \\
 & \underline{x_1^2x_3} \arrow[r, "\overline{\pi}_1"'] & \underline{x_2^2x_3}+x_1x_2x_3 \arrow[ur, "\overline{\pi}_2"'] &
\end{tikzcd}$$

The leading terms (underlined) of 1, $\overline{\pi}_1(x_1)$, $\overline{\pi}_2\overline{\pi}_1(x_1)$, $\overline{\pi}_2(x_1x_2)$, $\overline{\pi}_1\overline{\pi}_2(x_1x_2)$, and $\overline{\pi}_1\overline{\pi}_2\overline{\pi}_1(x_1^2x_2)$ are precisely the descent monomials 1, x_2, x_3, x_1x_3, x_2x_3, and $x_2x_3^2$ of the six permutations in $\mathfrak{S}_3$.

When $n=4$ and $\alpha=121$, one has $x_{D(\alpha)}=x_1^2x_2x_3$, and the five Demazure atoms $\overline{\pi}_w x_{D(\alpha)}$ with $D(w)=D(\alpha)$ are given below, whose leading terms (underlined) are the descent monomials of the corresponding permutations w.

$$\begin{tikzcd}
 & \underline{x_1x_2^2x_4} \arrow[d, "\overline{\pi}_2"] & \\
 & \underline{x_1x_3^2x_4}+x_1x_2x_3x_4 \arrow[dl, "\overline{\pi}_1"] \arrow[dr, "\overline{\pi}_3"] & \\
\underline{x_2x_3^2x_4} \arrow[dr, "\overline{\pi}_3"] & & \underline{x_1x_3x_4^2} \arrow[dl, "\overline{\pi}_1"'] \\
 & \underline{x_2x_3x_4^2} &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & 2{\cdot}14{\cdot}3 \arrow[d, "s_2"] & \\
 & 3{\cdot}14{\cdot}2 \arrow[dl, "s_1"] \arrow[dr, "s_3"] & \\
3{\cdot}24{\cdot}1 \arrow[dr, "s_3"] & & 4{\cdot}13{\cdot}2 \arrow[dl, "s_1"'] \\
 & 4{\cdot}23{\cdot}1 &
\end{tikzcd}$$

Proof. We prove the first assertion by induction on $\ell(w)$. It is trivial when $\ell(w)=0$. Assume $w=s_ju$ for some $j\in[n-1]$ and some $u\in\mathfrak{S}_n$ with $\ell(u)<\ell(w)$. Since

$D(u) \subseteq D(w) \subseteq D(\alpha)$, one has

$$\overline{\pi}_u x_{D(\alpha)} = u x_{D(\alpha)} + \sum_{x^d \prec x_{D(\alpha)}} c_d x^d, \quad c_d \in \mathbb{Z}. \tag{3.4}$$

It follows from (3.3) that

$$\overline{\pi}_j(x^d) = \sum_{x^e \preceq x^d} a_e x^e, \quad a_e \in \mathbb{Z}. \tag{3.5}$$

Observe that the degree of x_k in $u x_{D(\alpha)}$ is

$$r_k = \#\{i \in D(\alpha) : u^{-1}(k) \le i\}.$$

It follows from $\ell(s_j u) > \ell(u)$ that $u^{-1}(j) < u^{-1}(j+1)$ and thus $r_j \ge r_{j+1}$. Since $(s_j u)^{-1}(j+1) < (s_j u)^{-1}(j)$, there exists an $i \in D(s_j u) \subseteq D(\alpha)$ such that

$$u^{-1}(j) = (s_j u)^{-1}(j+1) \le i < (s_j u)^{-1}(j) = u^{-1}(j+1).$$

Thus $r_j > r_{j+1}$. It follows from (3.3) that

$$\overline{\pi}_j(u x_{D(\alpha)}) = s_j u x_{D(\alpha)} + \sum_{x^e \prec x_{D(\alpha)}} b_e x^e, \quad b_e \in \mathbb{Z}. \tag{3.6}$$

Combining (3.4), (3.5), and (3.6) one obtains the first assertion.

If $D(w) = D(\alpha)$ then $w x_{D(\alpha)}$ is the descent monomial of w. Conversely, assume $w x_{D(\alpha)}$ equals the descent monomial of some $u \in W$, i.e.

$$\prod_{i \in D(\alpha)} x_{w(1)} \cdots x_{w(i)} = \prod_{j \in D(u)} x_{u(1)} \cdots x_{u(j)}.$$

Let $D(\alpha) = \{i_1, \ldots, i_k\}$ and $D(u) = \{j_1, \ldots, j_t\}$. Comparing the variables absent on both sides of the above equality, one sees that $i_k = j_t$ and $w(i) = u(i)$ for $i = i_k+1, \ldots, n$. Repeating this argument for the variables appearing exactly m times, $m = 1, 2, \ldots$, one sees that $D(\alpha) = D(u)$ and $w = u$. □

Remark 3.1.3. Using the combinatorial formula by Mason [46] for the Demazure atoms, one can check that $\overline{\pi}_{w_0(\alpha)} x_{D(\alpha)}$ and $\overline{\pi}_{w_1(\alpha)} x_{D(\alpha)}$ are precisely the descent monomials of $w_0(\alpha)$ and $w_1(\alpha)$.

Lemma 3.1.4. *For any weak composition* $d = (d_1, \ldots, d_n)$*, let* $\sigma = \sigma(d)$*,* $\gamma = \gamma(d)$*,* $\mu = \mu(d)$*. If* $c_\beta \in \mathbb{Z}$ *for all* $\beta \prec \gamma$ *then*

$$m_\mu \cdot \left(x^\gamma + \sum_{\beta \prec \gamma} c_\beta x^\beta \right) = x^d + \sum_{x^e <_{ts} x^d} b_e x^e, \quad b_e \in \mathbb{Z}.$$

Proof. Since we already have (3.2), it suffices to show that $w\mu + \beta \prec d$ for all permutations w in $\mathfrak{S}_n$ and all $\beta \prec \gamma$. Given a weak composition α, let α_i be its i-th part. Since $\sigma\mu$ and $\sigma\gamma$ are both weakly decreasing, one has $\lambda(\mu)_i + \lambda(\gamma)_i = \lambda(d)_i$ for all $i = 1, \ldots, n$. Since $\beta \prec \gamma$, there exists a unique integer k such that $\lambda(\beta)_i = \lambda(\gamma)_i$ for $i = 1, \ldots, k-1$, and $\lambda(\beta)_k < \lambda(\gamma)_k$. Then for all $i \in [k-1]$,

$$\begin{aligned} \lambda(w\mu + \beta)_i &\leq \lambda(\mu)_i + \lambda(\beta)_i \\ &= \lambda(\mu)_i + \lambda(\gamma)_i \\ &= \lambda(d)_i \end{aligned}$$

and

$$\begin{aligned} \lambda(w\mu + \beta)_k &\leq \lambda(\mu)_k + \lambda(\beta)_k \\ &< \lambda(\mu)_k + \lambda(\gamma)_k \\ &= \lambda(d)_k. \end{aligned}$$

Therefore $w\mu + \beta \prec d$ and we are done. $\square$

Lemma 3.1.5. *The coinvariant algebra* $\mathbb{F}[X]/(\mathbb{F}[X]_+^{\mathfrak{S}_n})$ *has an* $\mathbb{F}$*-basis given by the set* $\{f_w : w \in \mathfrak{S}_n\}$*, if*

$$f_w = w x_{D(w)} + \sum_{x^d \prec x_{D(\alpha)}} c_d x^d, \quad c_d \in \mathbb{F}, \quad \forall w \in \mathfrak{S}_n.$$

Proof. Given a weak composition $d = (d_1, \ldots, d_n)$, let $\gamma = \gamma(d)$, $\mu = \mu(d)$, and $\sigma = \sigma(d)$. Then x^γ is the descent monomial of σ^{-1}. By Lemma 3.1.4,

$$m_\mu f_{\sigma^{-1}} = x^d + \sum_{x^e <_{ts} x^d} b_e x^e.$$

Hence $\{m_\mu f_w : \mu = (\mu_1 \geq \cdots \geq \mu_n \geq 0),\ w \in \mathfrak{S}_n\}$ is triangularly related to the set of all monomials x^d, and thus an $\mathbb{F}$-basis for $\mathbb{F}[X]$. It follows that $\{f_w : w \in \mathfrak{S}_n\}$ is an $\mathbb{F}[X]^{\mathfrak{S}_n}$-basis for $\mathbb{F}[X]$ and gives an $\mathbb{F}$-basis for $\mathbb{F}[X]/(\mathbb{F}[X]_+^{\mathfrak{S}_n})$. $\square$

Theorem 3.1.6. *The coinvariant algebra* $\mathbb{F}[X]/(\mathbb{F}[X]_+^{\mathfrak{S}_n})$ *has an* $\mathbb{F}$*-basis given by the set* $\left\{\overline{\pi}_w x_{D(w)} : w \in \mathfrak{S}_n\right\}$ *and an* $H_n(0)$*-module decomposition*

$$\mathbb{F}[X]/(\mathbb{F}[X]_+^{\mathfrak{S}_n}) = \bigoplus_{\alpha \models n} H_n(0) \cdot \overline{\pi}_{w_0(\alpha)} x_{D(\alpha)}$$

where $H_n(0) \cdot \overline{\pi}_{w_0(\alpha)} x_{D(\alpha)}$ *has an* $\mathbb{F}$*-basis* $\left\{\overline{\pi}_w x_{D(\alpha)} : w \in [w_0(\alpha), w_1(\alpha)]\right\}$*, and is isomorphic to the projective indecomposable* $H_n(0)$*-module* $\mathbf{P}_\alpha$*, for all* $\alpha \models n$*. Consequently,* $\mathbb{F}[X]/(\mathbb{F}[X]_+^{\mathfrak{S}_n})$ *is isomorphic to the regular representation of* $H_n(0)$*.*

Proof. By Lemma 3.1.1 and Lemma 3.1.5, the set $\{\overline{\pi}_w x_{D(w)} : w \in \mathfrak{S}_n\}$ gives a basis for $\mathbb{F}[X]/(\mathbb{F}[X]_+^{\mathfrak{S}_n})$. For any permutation u in $\mathfrak{S}_n$, one sees from the relations of $\overline{\pi}_i$ that $\overline{\pi}_u \overline{\pi}_{w_0(\alpha)} = \pm \overline{\pi}_w$ for some $w \geq w_0(\alpha)$ in the left weak order, which implies $D(w) \supseteq D(\alpha)$. If there exists $j \in D(w) \setminus D(\alpha)$, then

$$\overline{\pi}_w x_{D(\alpha)} = \overline{\pi}_{ws_j} \overline{\pi}_j x_{D(\alpha)} = 0.$$

Hence $H_n(0) \cdot \overline{\pi}_{w_0(\alpha)} x_{D(\alpha)}$ is spanned by $\{\overline{\pi}_w x_{D(\alpha)} : w \in [w_0(\alpha), w_1(\alpha)]\}$, which must be an $\mathbb{F}$-basis since it is a subset of a linearly independent set. Sending $\overline{\pi}_w x_{D(\alpha)}$ to $\overline{\pi}_w \pi_{w_0(\alpha^c)}$ for all $w \in [w_0(\alpha), w_1(\alpha)]$ gives an isomorphism between $H_n(0) \cdot \overline{\pi}_{w_0(\alpha)} x_{D(\alpha)}$ and $\mathbf{P}_\alpha$. □

Remark 3.1.7. (i) This theorem and its proof are valid when $\mathbb{F}$ is replaced with $\mathbb{Z}$.
(ii) By Remark 3.1.3, the cyclic generators $\overline{\pi}_{w_0(\alpha)} x_{D(\alpha)}$ for the indecomposable summands of the coinvariant algebra are precisely the descent monomials $w_0(\alpha) x_{D(\alpha)}$.

As a graded $\mathfrak{S}_n$-mdoule, the coinvariant algebra $\mathbb{C}[X]/(\mathbb{C}[X]_+^{\mathfrak{S}_n})$ has graded Frobenius characteristic given by the following result, which is due to G. Lusztig (unpublished) and independently to R. Stanley [54, Proposition 4.11].

Theorem 3.1.8 (Lusztig-Stanley)**.** *The graded Frobenius characteristic of the coinvariant algebra of* $\mathfrak{S}_n$ *is*

$$\mathrm{ch}_t\left(\mathbb{C}[X]/(\mathbb{C}[X]_+^{\mathfrak{S}_n})\right) = \sum_{\lambda \vdash n} \sum_{\tau \in \mathrm{SYT}(\lambda)} t^{\mathrm{maj}(\tau)} s_\lambda \overset{(*)}{=\!=} \widetilde{H}_{1^n}(x;t)$$

where t *keeps track of the usual polynomial degree grading, and* $\widetilde{H}_{1^n}(x;t)$ *is the modified Hall-Littlewood symmetric function of the partition* 1^n*.*

Remark 3.1.9. The equality $(*)$ is a special case of Theorem 3.4.1. One can also see it by using the charge formula of Lascoux and Schützenberger [43].

We have an analogous result for the $H_n(0)$-action on the coinvariant algebra.

Corollary 3.1.10. *(i) The bigraded characteristic of the coinvariant algebra is*

$$\mathrm{Ch}_{q,t}\left(\mathbb{F}[X]/(\mathbb{F}[X]_+^{\mathfrak{S}_n})\right) = \sum_{w\in\mathfrak{S}_n} t^{\mathrm{maj}(w)} q^{\mathrm{inv}(w)} F_{D(w^{-1})}$$

where q keeps track of the grading from the length filtration of the regular representation of $H_n(0)$, and t keeps track of the polynomial degree grading.
(ii) The degree graded quasisymmetric characteristic of the coinvariant algebra is

$$\mathrm{Ch}_t\left(\mathbb{F}[X]/(\mathbb{F}[X]_+^{\mathfrak{S}_n})\right) = \sum_{\alpha\models n} r_\alpha(t) F_\alpha = \sum_{\alpha\models n}\sum_{\tau\in\mathrm{SYT}(\alpha)} t^{\mathrm{maj}(\tau)} F_\alpha.$$

(iii) The quasisymmetric function in (ii) is actually symmetric and equals

$$\sum_{\lambda\vdash n}\begin{bmatrix} n \\ \lambda\end{bmatrix}_t m_\lambda = \sum_{\lambda\vdash n}\sum_{\tau\in\mathrm{SYT}(\lambda)} t^{\mathrm{maj}(\tau)} s_\lambda = \sum_{\lambda\vdash n} t^{n(\lambda)}\frac{[n]!_t}{\prod_{u\in\lambda}[h_u]_t} s_\lambda = \widetilde{H}_{1^n}(x;t)$$

where h_u is the hook length of the box u in the Young diagram of the partition λ and $n(\lambda) := \lambda_2 + 2\lambda_3 + 3\lambda_4 + \cdots$ if $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ (c.f. Stanley [56, §7.21]).

Proof. Given a composition α of n, the $H_n(0)$-module $H_n(0)\cdot\overline{\pi}_{w_0(\alpha)} x_{D(\alpha)}$ is isomorphic to projective indecomposable $\mathbf{P}_\alpha$, with isomorphism given by the bijection between their bases:

$$\overline{\pi}_w x_{D(\alpha)} \leftrightarrow \overline{\pi}_w \pi_{w_0(\alpha^c)}, \quad \forall w \in [w_0(\alpha), w_1(\alpha)].$$

For any $w \in [w_0(\alpha), w_1(\alpha)]$, the element $\overline{\pi}_w x_{D(\alpha)}$ has homogeneous degree $\mathrm{maj}(w)$, and

$$\overline{\pi}_w \pi_{w_0(\alpha^c)} = \sum_{u\in\mathfrak{S}_\alpha} \overline{\pi}_{wu}$$

is the sum of $\overline{\pi}_w$ and other elements of lager lengths. Hence (i) follows from Theorem 3.1.6.

It follows from (i) that the degree graded multiplicity of a simple $H_n(0)$-module $\mathbf{C}_\alpha$ in the coinvariant algebra is

$$r_\alpha(t) = \sum_{\substack{w\in\mathfrak{S}_n:\\ D(w^{-1})=D(\alpha)}} t^{\mathrm{maj}(w)} = \sum_{\substack{w\in\mathfrak{S}_n:\\ D(w)=D(\alpha)}} t^{\mathrm{maj}(w^{-1})} = \sum_{\tau\in\mathrm{SYT}(\alpha)} t^{\mathrm{maj}(\tau)}.$$

Then

$$\mathrm{Ch}_t\left(\mathbb{F}[X]/(\mathbb{F}[X]_+^{\mathfrak{S}_n})\right) = \sum_{\alpha\models n} r_\alpha(t)F_\alpha = \sum_{\alpha\models n} \begin{bmatrix} n \\ \alpha \end{bmatrix}_t M_\alpha = \sum_{\lambda\vdash n} \begin{bmatrix} n \\ \lambda \end{bmatrix}_t m_\lambda \in \mathrm{Sym}[t].$$

Given a partition μ of n, we have

$$\begin{bmatrix} n \\ \mu \end{bmatrix}_t = \sum_{\substack{w\in\mathfrak{S}_n: \\ D(w)\subseteq D(\mu)}} t^{\mathrm{maj}(w^{-1})} = \sum_{w\in\mathfrak{S}(\mu)} t^{\mathrm{maj}(w^{-1})}$$

where $\mathfrak{S}(\mu)$ is the set of all permutations of the multiset of type μ. For example, $w = 3561247$ corresponds to

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 3 & 3 \\ 3 & 5 & 6 & 1 & 2 & 4 & 7 \end{pmatrix} \in \mathfrak{S}(332).$$

Applying RSK to $w \in \mathfrak{S}(\mu)$ gives a pair (P, Q) of Young tableaux P and Q of the same shape (say λ), where P is standard, and Q is semistandard of type μ. It is well-known that the descents of w^{-1} are precisely the descents of P; see e.g. Schützenberger [51]. Hence

$$\begin{aligned} \mathrm{Ch}_t\left(\mathbb{F}[X]/(\mathbb{F}[X]_+^{\mathfrak{S}_n})\right) &= \sum_{\lambda\vdash n}\sum_{P\in\mathrm{SYT}(\lambda)} t^{\mathrm{maj}(P)} \sum_{\mu\vdash n} K_{\lambda\mu} m_\mu \\ &= \sum_{\lambda\vdash n} t^{n(\lambda)} \frac{[n]!_t}{\prod_{u\in\lambda}[h_u]_t} s_\lambda. \end{aligned}$$

Here the last equality follows from the the q-hook length formula (see e.g. Stanley [56, Proposition 7.21.5]) and (2.8). □

3.2 Coinvariant algebra of Weyl groups

The results in the previous section can be generalized to the action of the 0-Hecke algebra of a Weyl group W on the Laurent ring $\mathbb{F}[\Lambda]$ of the weight lattice Λ of W. The readers are referred to Humphreys [36] for details on Weyl groups and weight theory.

Demazure's character formula [16] expresses the character of the highest weight modules over a semisimple Lie algebra using the *Demazure operators* π_i on the group ring $\mathbb{F}[\Lambda]$ of the weight lattice Λ. Write e^λ for the element in $\mathbb{F}[\Lambda]$ corresponding to the weight $\lambda \in \Lambda$. Suppose that $\gamma_1, \ldots, \gamma_r$ are the simple roots[1], $s_1, \ldots, s_r$ are the simple reflections, and $\lambda_1, \ldots, \lambda_r$ are the fundamental weights. Then

$$\mathbb{F}[\Lambda] = \mathbb{F}[z_1, \ldots, z_r, z_1^{-1}, \ldots, z_r^{-1}]$$

where $z_i = e^{\lambda_i}$. The *Demazure operators* are defined by

$$\pi_i = \frac{f - e^{-\gamma_i} s_i(f)}{1 - e^{-\gamma_i}}, \quad \forall f \in \mathbb{F}[\Lambda].$$

It follows that

$$\pi_i(e^{\lambda}) = \begin{cases} e^{\lambda} + e^{\lambda - \gamma_i} + \cdots + e^{s_i \lambda}, & \text{if } \langle \lambda, \gamma_i \rangle \geq 0, \\ 0, & \text{if } \langle \lambda, \gamma_i \rangle = -1, \\ -e^{\lambda + \gamma_i} - \cdots - e^{s_i \lambda - \gamma_i}, & \text{if } \langle \lambda, \gamma_i \rangle < -1. \end{cases} \tag{3.7}$$

Here $\langle \lambda, \gamma_i \rangle = 2(\lambda, \gamma_i)/(\gamma_i, \gamma_i)$ with $(-,-)$ being the standard inner product. See, for example, Kumar [41]. The Demazure operators satisfy $s_i \pi_i = \pi_i$, $\pi_i^2 = \pi_i$, and the braid relations [16, §5.5] $\pi_i \pi_{i+1} \pi_i = \pi_{i+1} \pi_i \pi_{i+1}$ and $\pi_i \pi_j = \pi_j \pi_i$ if $|i - j| > 1$. Hence the 0-Hecke algebra $H_W(0)$ of the Weyl group W acts on $\mathbb{F}[\Lambda]$ by π_i, or equivalently by $\overline{\pi}_i := \pi_i - 1$. It is clear that $\pi_i f = f$ if and only if $s_i f = s_i f$ for all $i \in [r]$ and $f \in \mathbb{F}[\Lambda]$.

Using the Stanley-Reisner ring of the Coxeter complex of W (defined in §5.1), Garsia and Stanton [27] showed that

$$\mathbb{F}[\Lambda]^W = \mathbb{F}[a_1, \ldots, a_r]$$

where

$$a_i = \sum_{w \in W/W_{i^c}} e^{w \lambda_i}, \quad (i^c = [r] \setminus \{i\})$$

and $\mathbb{F}[\Lambda]$ has a free basis over $\mathbb{F}[\Lambda]^W$, which consists of the *descent monomials*

$$z_w := \prod_{i \in D(w)} e^{w \lambda_i}, \quad \forall w \in W.$$

See also Steinberg [57]. If we write $\lambda_I = \sum_{i \in I} \lambda_i$ for all $I \subseteq [r]$, then $z_w = w e^{\lambda_{D(w)}}$. The basis $\{z_w : w \in W\}$ induces an $\mathbb{F}$-basis for $\mathbb{F}[\Lambda]/(a_1, \ldots, a_r)$. The $H_W(0)$-action on $\mathbb{F}[\Lambda]$ is $\mathbb{F}[\Lambda]^W$-linear, hence inducing an $H_W(0)$-action on $\mathbb{F}[\Lambda]/(a_1, \ldots, a_r)$.

[1] We use γ to denote roots because α is used for compositions.

We order the weights by $\lambda \leq \mu$ if $\mu - \lambda$ is a nonnegative linear combination of simple roots. Every monomial is $\mathbb{F}[\Lambda]$ is of the form $m = e^\lambda$ for some weight λ. By Humphreys [36], there exists a unique dominant weight μ such that $\mu = w\lambda$ for some w in W, and we have $\lambda \leq \mu$. Write $[m]_+ = [\lambda]_+ := \mu$ and call this dominant weight μ the *shape* of the monomial m or the weight λ.

For every monomial m of shape λ, Garsia and Stanton [27, proof of Theorem 9.4] showed that

$$m - \sum_{\substack{d \in \mathbb{Z}^r,\ w \in W: \\ \lambda_d + \lambda_{D(w)} = \lambda}} c_{d,w} a_1^{d_1} \cdots a_r^{d_r} z_w \tag{3.8}$$

is a linear combination of monomials whose shape is strictly less than λ, where $c_{d,w} \in \mathbb{Z}$ and $\lambda_d = d_1\lambda_1 + \cdots + d_r\lambda_r$. It follows by induction that the descent monomials z_w form an $\mathbb{F}[\Lambda]^W$-basis for $\mathbb{F}[\Lambda]$.

Lemma 3.2.1. *Suppose that γ is a simple root and λ is a weight such that $\langle \lambda, \gamma \rangle \geq 0$. If $0 \leq k \leq \langle \lambda, \gamma \rangle$ then $[\lambda - k\gamma]_+ \leq [\lambda]_+$, and the equality holds if and only if $k = 0$ or $\langle \lambda, \gamma \rangle$.*

Proof. Let $\mu = \lambda - k\gamma$. If $k = 0$ or $\langle \lambda, \gamma \rangle$, then $\mu = \lambda$ or $s_\gamma\lambda$, and thus $[\mu]_+ = [\lambda]_+$ in either case. Assume $0 < k < \langle \lambda, \gamma \rangle$ below, and let $w\lambda$ and $u\mu$ be dominant for some w and u in W.

If $u\gamma > 0$ then $u\mu = uw^{-1}(w\lambda) - ku\gamma < uw^{-1}(w\lambda) \leq w\lambda$.

If $u\gamma < 0$ then

$$\begin{aligned} u\mu &= us_\gamma \cdot s_\gamma\lambda - ku\gamma \\ &= us_\gamma(\lambda - \langle \lambda, \gamma \rangle \gamma) - ku\gamma \\ &= us_\gamma\lambda + (\langle \lambda, \gamma \rangle - k)u\gamma \\ &< us_\gamma\lambda \leq w\lambda. \end{aligned}$$

This completes the proof. $\square$

Lemma 3.2.2. *Given a composition α of $r+1$, let $\lambda_\alpha = \lambda_{D(\alpha)}$ and $z_\alpha = e^{\lambda_\alpha}$. If $w \in W$ has $D(w) \subseteq D(\alpha)$, then*

$$\overline{\pi}_w z_\alpha = e^{w\lambda_\alpha} + \sum_{[\lambda]_+ < \lambda_\alpha} c_\lambda e^\lambda, \quad c_\lambda \in \mathbb{Z}.$$

Moreover, $e^{w\lambda_\alpha}$ *is a descent monomial if and only if* $D(w) = D(\alpha)$*.*

Proof. We prove the first assertion by induction on $\ell(w)$. If $\ell(w) = 0$ then we are done; otherwise $w = s_j u$ for some $j \in [r]$ and for some u with $\ell(u) < \ell(w)$. Since $D(u) \subseteq D(w) \subseteq D(\alpha)$, one has

$$\overline{\pi}_u z_\alpha = e^{u\lambda_\alpha} + \sum_{[\lambda]_+ < \lambda_\alpha} c_\lambda e^\lambda, \quad c_\lambda \in \mathbb{Z}.$$

Applying Lemma 3.2.1 to (3.7) (if the simple root γ_j satisfies $\langle \lambda, \gamma_j \rangle \leq 0$ then $\langle s_j\lambda, \gamma_j \rangle \geq 0$), one sees that

$$\overline{\pi}_j(e^\lambda) = \sum_{[\mu]_+ \leq [\lambda]_+} a_\mu e^\mu, \quad a_\mu \in \mathbb{Z}.$$

If we can show $\langle u\lambda_\alpha, \gamma_j \rangle > 0$, then applying Lemma 3.2.1 to the first case of (3.7) one has

$$\overline{\pi}_j e^{u\lambda_\alpha} = e^{s_j u\lambda_\alpha} + \sum_{[\mu]_+ < \lambda_\alpha} b_\mu e^\mu, \quad b_\mu \in \mathbb{Z}.$$

Combining these equations one obtains

$$\overline{\pi}_w z_\alpha = e^{w\lambda_\alpha} + \sum_{[\mu]_+ < \lambda_\alpha} b_\mu e^\mu + \sum_{[\lambda]_+ < \lambda_\alpha} c_\lambda \sum_{[\mu]_+ \leq [\lambda]_+} a_\mu e^\mu,$$

which gives the desired result.

Now we prove $\langle u\lambda_\alpha, \gamma_j \rangle > 0$. In fact, since $\ell(s_j u) > \ell(u)$, one has $u^{-1}(\gamma_j) > 0$, i.e.

$$u^{-1}(\gamma_j) = \sum_{i=1}^{r} m_i \gamma_i$$

for some nonnegative integers m_i. Applying $s_j u$ to both sides one gets

$$0 > -\gamma_j = \sum_{i=1}^{r} m_i s_j u(\gamma_i).$$

By the hypothesis $D(s_j u) \subseteq D(\alpha)$, if $i \notin D(\alpha)$ then $s_j u(\gamma_i) > 0$. This forces $m_i > 0$ for some $i \in D(\alpha)$, and thus

$$\langle u\lambda_\alpha, \gamma_j \rangle = \langle \lambda_\alpha, u^{-1}\gamma_j \rangle = \sum_{i \in D(\alpha)} m_i > 0.$$

Finally we consider when $e^{w\lambda_\alpha}$ is a descent monomial. If $D(w) = D(\alpha)$ then it is just the descent monomial of w. Conversely, if it is a descent monomial of some $u \in W$

then $w\lambda_\alpha = u\lambda_{D(u)}$. Since λ_α and $\lambda_{D(u)}$ are both in the fundamental Weyl chamber, the above equality implies that $\lambda_\alpha = \lambda_{D(u)}$ and $u^{-1}w$ is a product of simple reflections which all fix λ_α ([36, Lemma 10.3B]), *i.e.*

$$w = us_{j_1}\cdots s_{j_k}, \quad j_1,\ldots,j_k \notin D(u) = D(\alpha).$$

Since $D(w) \subseteq D(\alpha) = D(u)$, none of $s_{j_1},\ldots,s_{j_k}$ is a descent of w, and thus it follows from the deletion property of W that w is a subword of some reduced expression of u, i.e. $w \le u$ in Bruhat order. Similarly, it follows from $u = ws_{j_k}\cdots s_{j_1}$ that $u \le w$ in Bruhat order. Thus $u = w$. □

Theorem 3.2.3. *The coinvariant algebra $\mathbb{F}[\Lambda]/(a_1,\ldots,a_r)$ has an $\mathbb{F}$-basis given by $\left\{\overline{\pi}_w e^{\lambda_{D(w)}} : w \in W\right\}$ and an $H_W(0)$-module decomposition*

$$\mathbb{F}[\Lambda]/(a_1,\ldots,a_r) = \bigoplus_{\alpha\models r+1} H_W(0)\cdot\overline{\pi}_{w_0(\alpha)}z_\alpha$$

where each direct summand $H_W(0)\cdot\overline{\pi}_{w_0(\alpha)}z_\alpha$ has an $\mathbb{F}$-basis

$$\{\overline{\pi}_w z_\alpha : w \in [w_0(\alpha), w_1(\alpha)]\}$$

and is isomorphic to the projective indecomposable $H_W(0)$-module $\mathbf{P}_\alpha$. Consequently, $\mathbb{F}[\Lambda]/(a_1,\ldots,a_r)$ is isomorphic to the regular representation of $H_W(0)$.

Proof. By Lemma 3.2.2, if one replaces the descent monomial z_w with the Demazure atom $\overline{\pi}_w e^{\lambda_{D(w)}}$ in (3.8), the extra terms produced are of the form

$$c_{d,w}a_1^{d_1}\cdots a_r^{d_r}e^\mu$$

where $d = (d_1,\ldots,d_r)$ and μ are weak compositions, and w is an element in W, satisfying $\lambda_d + \lambda_{D(w)} = \lambda$ and $[\mu] < \lambda_{D(w)}$. By the definition of $a_1,\ldots,a_r$, one expands each term above as a linear combination of the monomials

$$e^{d_1w_1\lambda_1}\cdots e^{d_rw_r\lambda_r}e^\mu, \quad w_i \in W/W_{i^c}.$$

There exists $w \in W$ such that

$$\begin{aligned}
& [\mu + d_1w_1\lambda_1 + \cdots + d_rw_r\lambda_r] \\
= \ & w(\mu + d_1w_1\lambda_1 + \cdots + d_rw_r\lambda_r) \\
\le \ & [\mu] + d_1\lambda_1 + \cdots + d_r\lambda_r \\
< \ & \lambda_{D(w)} + \lambda_d = \lambda.
\end{aligned}$$

By induction on the shapes of the monomials, one shows that the Demazure atoms $\overline{\pi}_w \lambda_{D(w)}$ for all w in W form an $\mathbb{F}[\Lambda]^W$-basis for $\mathbb{F}[\Lambda]$, giving an $\mathbb{F}$-basis for the coinvariant algebra $\mathbb{F}[\Lambda]/(a_1, \ldots, a_r)$.

Let $\alpha \models r+1$. For any u in W, using $\overline{\pi}_i^2 = -\overline{\pi}_i$ one shows by induction that $\overline{\pi}_u \overline{\pi}_{w_0(\alpha)} = \overline{\pi}_w$ for some $w \geq w_0(\alpha)$ in the (left) weak order, which implies $D(w) \supseteq D(\alpha)$. On the other hand, if there exists $j \in D(w) \setminus D(\alpha)$, then $\overline{\pi}_w z_\alpha = 0$ since $\overline{\pi}_j z_\alpha = 0$ by (3.7). Hence $H_W(0) \cdot \overline{\pi}_{w_0(\alpha)} z_\alpha$ has a basis $\{\overline{\pi}_w z_\alpha : w \in [w_0(\alpha), w_1(\alpha)]\}$, and is isomorphic to $\mathbf{P}_\alpha$ via $\overline{\pi}_w z_\alpha \mapsto \overline{\pi}_w \pi_{w_0(\alpha^c)}$ for all $w \in [w_0(\alpha), w_1(\alpha)]$. $\square$

Remark 3.2.4. Garsia and Stanton [27] pointed out a way to reduce the descent monomials in $\mathbb{F}[\Lambda]$ to the descent monomials in $\mathbb{F}[X]$ for type A. However, it does not give Theorem 3.1.6 directly from Theorem 3.2.3; instead, one should consider the Demazure operators on $\mathbb{F}[X(T)]$ where $X(T)$ is the character group of the subgroup T of diagonal matrices in $GL(n, \mathbb{F})$.

3.3 Flag varieties

In this section we study the action of the 0-Hecke algebras on the (complete) flag varieties. Assume $\mathbb{F}$ is a finite field $\mathbb{F}_q$ of q elements, where q is a power of a prime p, throughout this section.

Let G be a finite group of Lie type over $\mathbb{F}_q$, with Borel subgroup B and Weyl group W. Assume that W is generated by simple reflections $s_1, \ldots, s_r$. Every composition α of $r+1$ corresponds to a *parabolic subgroup* $P_\alpha := BW_{D(\alpha)^c}B$ of G. The *partial flag variety* $1_{P_\alpha}^G$ is the induction of the trivial representation of P_α to G, i.e. the $\mathbb{F}_q$-span of all right P_α-cosets in G. Taking $\alpha = 1^{r+1}$ we have the *(complete) flag variety* 1_B^G.

For type A, one has $G = GL(n, \mathbb{F}_q)$, and if $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ is a composition of n, then P_α is the group of all block upper triangular matrices with invertible diagonal blocks of sizes $\alpha_1, \ldots, \alpha_\ell$. Using the action of G on the vector space $V = \mathbb{F}_q^n$, one can identify $1_{P_\alpha}^G$ with the $\mathbb{F}_q$-span of all partial flags of subspaces $0 \subset V_1 \subset \cdots \subset V_\ell = V$ satisfying $\dim V_i = \alpha_i$ for $i = 1, \ldots, \ell$; in particular, 1_B^G is the $\mathbb{F}_q$-span of all complete flags of V.

3.3.1 0-Hecke algebra action on 1_B^G

Given a subset $H \subseteq G$, let

$$\overline{H} = \sum_{h \in H} h \qquad \text{inside } \mathbb{Z}G.$$

Then the right $\mathbb{Z}G$-module $\overline{B} \cdot \mathbb{Z}G$ is isomorphic to the induction of the trivial representation of B to G. By work of Kuhn [40], the endomorphism ring $\mathrm{End}_{\mathbb{Z}G}(\overline{B} \cdot \mathbb{Z}G)$ has a basis $\{f_w : w \in W\}$, with f_w given by

$$f_w(\overline{B}) = \overline{BwB} = \overline{U}_w w \overline{B}.$$

Here U_w is the product of the root subgroups of those positive roots which are sent to negative roots by w^{-1} (see [18, Proposition 1.7]). The endomorphism ring $\mathrm{End}_{\mathbb{Z}G}(\overline{B} \cdot \mathbb{Z}G)$ is isomorphic to the Hecke algebra $H_W(q)$ of W with parameter $q = |U_{s_i}|$, since the relations satisfied by $\{f_w : w \in W\}$ are the same as those satisfied by the standard basis $\{T_w : w \in W\}$ for $H_W(q)$.

By extending scalars from $\mathbb{Z}$ to $\mathbb{F} = \mathbb{F}_q$, we obtain a G-equivariant action of the 0-Hecke algebra $H_W(0)$ on 1_B^G by

$$\overline{\pi}_w(\overline{B}g) := \overline{BwB}g, \quad \forall g \in G,\ \forall w \in W.$$

We will use left cosets in the next subsection, and in that case there is a similar right $H_W(0)$-action.

Given a finite dimensional filtered $H_W(0)$-module Q and a composition α of $r+1$, define Q_α to be the $\mathbb{F}$-subspace of the elements in Q that are annihilated by $\overline{\pi}_j$ for all $j \notin D(\alpha)$, i.e.

$$Q_\alpha := \bigcap_{j \in D(\alpha)^c} \ker \overline{\pi}_j.$$

The next lemma gives the simple composition factors of Q by inclusion-exclusion.

Lemma 3.3.1. *Given a finite dimensional $H_W(0)$-module Q and a composition α of $r+1$, the multiplicity of the simple $H_W(0)$-module $\mathbf{C}_\alpha$ among the composition factors of Q is*

$$c_\alpha(Q) = \sum_{\beta \preccurlyeq \alpha} (-1)^{\ell(\alpha)-\ell(\beta)} \dim_{\mathbb{F}_q}(Q_\beta).$$

Proof. Let $0 = Q_0 \subset Q_1 \subset \cdots \subset Q_k = Q$ be a composition series. We induct on the composition length k. The case $k = 0$ is trivial. Assume $k > 0$ below.

Suppose $Q/Q' \cong \mathbf{C}_\gamma$ for some $\gamma \models r+1$, i.e. there exists an element z in $Q \setminus Q'$ satisfying

$$\overline{\pi}_i z \in \begin{cases} -z + Q', & \text{if } i \in D(\gamma), \\ Q', & \text{if } i \notin D(\gamma). \end{cases}$$

Let u be the longest element of the parabolic subgroup $W_{D(\gamma)^c}$ of W, and let

$$z' = \pi_u z = \sum_{w \in W_{D(\gamma)^c}} \overline{\pi}_w z.$$

Since $D(w) \subseteq D(\gamma)^c$ for all w in the above sum, we have $z' \in z + Q'$. Then any element in

$$Q = Q' \oplus \mathbb{F} z = Q' \oplus \mathbb{F} z'$$

can be written as $y + az'$ for some $y \in Q'$ and $a \in \mathbb{F}$. Since $D(u^{-1}) = D(\gamma)^c$, one has $\overline{\pi}_i z' = \overline{\pi}_i \pi_u z = 0$ for all $i \notin D(\gamma)$. Consider an arbitrary composition β of $r+1$.

If $\gamma \preccurlyeq \beta$ then for any $i \notin D(\beta)$ we must have $i \notin D(\gamma)$ and thus $\overline{\pi}_i(y + az') = \overline{\pi}_i y$. It follows that $\overline{\pi}_i(y+az') = 0$ if and only if $\overline{\pi}_i y = 0$, i.e. $Q_\beta = Q'_\beta \oplus \mathbb{F} z'$.

If $\gamma \not\preccurlyeq \beta$ then there exists $i \in D(\gamma) \setminus D(\beta)$. Using $z' \in z + Q'$ we have

$$\overline{\pi}_i(y + az') = \overline{\pi}_i y + a\overline{\pi}_i z' \in -az + Q'.$$

If $\overline{\pi}_i(y+az') = 0$ then $a = 0$. This implies $Q_\beta = Q'_\beta$.

It follows that

$$\begin{aligned} c_\alpha(Q) &= \sum_{\beta \preccurlyeq \alpha} (-1)^{\ell(\alpha)-\ell(\beta)} \dim_{\mathbb{F}_q}(Q_\beta) \\ &= \sum_{\beta \preccurlyeq \alpha} (-1)^{\ell(\alpha)-\ell(\beta)} \dim_{\mathbb{F}_q}(Q'_\beta) + \sum_{\gamma \preccurlyeq \beta \preccurlyeq \alpha} (-1)^{\ell(\alpha)-\ell(\beta)} \\ &= c_\alpha(Q') + \delta_{\alpha\gamma}. \end{aligned}$$

On the other hand, by induction hypothesis, the multiplicity of $\mathbf{C}_\alpha$ in the composition factors of Q is also $c_\alpha(Q') + \delta_{\alpha\gamma}$. Hence we are done. $\square$

Corollary 3.3.2. *(i) If Q is a finite dimensional graded $H_W(0)$-module and α is a composition of $r+1$, then the graded multiplicity of the simple $H_W(0)$-module $\mathbf{C}_\alpha$ among the composition factors of Q is*

$$c_\alpha(Q) = \sum_{\beta \preccurlyeq \alpha} (-1)^{\ell(\alpha)-\ell(\beta)} \mathrm{Hilb}(Q_\beta, t).$$

(ii) Let $W = \mathfrak{S}_n$. Then

$$\mathrm{Ch}_t(Q) = \sum_{\alpha \models r+1} \mathrm{Hilb}(Q_\alpha, t) M_\alpha.$$

Consequently, $\mathrm{Ch}_t(Q) \in \mathrm{QSym}[t]$ *is a symmetric function, i.e. it lies in* $\mathrm{Sym}[t]$, *if and only if*

$$\mathrm{Hilb}(Q_\alpha, t) = \mathrm{Hilb}(Q_\beta, t) \quad \textit{whenever } \beta \textit{ is a rearrangement of } \alpha.$$

Proof. Applying the previous lemma to each homogeneous component of Q one obtains (i). Then using inclusion-exclusion one obtains (ii). □

Remark 3.3.3. Lemma 3.3.1 and Corollary 3.3.2 hold for an arbitrary field $\mathbb{F}$.

Theorem 3.3.4. *The multiplicity of $\mathbf{C}_\alpha$ among the simple composition factors of 1_B^G is*

$$c_\alpha(1_B^G) = \sum_{w \in W : D(w^{-1}) = D(\alpha)} q^{\ell(w)}.$$

Proof. Let $\overline{B}g$ be an element in 1_B^G where $g \in \mathbb{F}G$. If it is annihilated by $\overline{\pi}_j$ for all $j \in D(\alpha)^c$, then $\overline{BwBg} = \overline{\pi}_w(\overline{B}g) = 0$ for all w with $D(w) \cap D(\alpha)^c \neq \emptyset$, and in particular, for all w in $W_{D(\alpha)^c} \setminus \{1\}$. Hence

$$\overline{B}g = \overline{BW_{D(\alpha)^c}B}g = \overline{P}_\alpha g \in 1_{P_\alpha}^G.$$

Conversely, $\overline{\pi}_j(\overline{P}_\alpha g) = \overline{\pi}_j \pi_{w_0(D(\alpha)^c)}(\overline{B}g) = 0$ for all $j \in D(\alpha)^c$. Therefore $(1_B^G)_\alpha = 1_{P_\alpha}^G$. Applying Lemma 3.3.1 to 1_B^G gives

$$\begin{aligned} c_\alpha(1_B^G) &= \sum_{\beta \preccurlyeq \alpha} (-1)^{\ell(\alpha)-\ell(\beta)} |P_\alpha \backslash G| \\ &= \sum_{\beta \preccurlyeq \alpha} (-1)^{\ell(\alpha)-\ell(\beta)} \sum_{w \in W : D(w^{-1}) \subseteq D(\alpha)} |U_w| \\ &= \sum_{w \in W : D(w^{-1}) = D(\alpha)} q^{\ell(w)}. \end{aligned}$$

□

Corollary 3.3.5. *If* $G = GL(n, \mathbb{F}_q)$ *then* $\mathrm{Ch}\left(1_B^G\right) = \widetilde{H}_{1^n}(x; q)$.

Proof. For $G = GL(n, \mathbb{F}_q)$ we have $\ell(w) = \mathrm{inv}(w)$ and thus equation (2.2) shows $c_\alpha(1_B^G) = r_\alpha(q)$. The result then follows from Corollary 3.1.10. $\square$

3.3.2 Decomposing the G-module 1_B^G by 0-Hecke algebra action

We consider the *homology representations* χ_q^α of G, which are the top homology of the *type-selected Tits buildings* of G, for all compositions $\alpha \models r+1$. To give the explicit definitions, assume in this subsection that $1_{P_\alpha}^G$ is the $\mathbb{F}_q$-span of *left* P_α-cosets in G. Then 1_B^G admits a right $H_W(0)$-action defined by $g\overline{B} \cdot \overline{\pi}_w = g\overline{BwB}$ for all $g \in G$ and $w \in W$. The left cosets gP_α for all $\alpha \models r+1$ form a poset under reverse inclusion, giving an (abstract) simplicial complex called the *Tits building* and denoted by $\Delta = \Delta(G, B)$. The type of a face gP_α is $\tau(gP_\alpha) = D(\alpha)$, and every chamber gB has exactly one vertex of each type, i.e. $\Delta(G, B)$ is *balanced*.

If β and γ are compositions of $r+1$ with $D(\beta) = D(\gamma) \setminus \{i\}$ for some $i \in D(\gamma)$, then we write $\beta \preccurlyeq_1 \gamma$ and $[\beta : \gamma] = (-1)^i$. The chain complex of the *type-selected subcomplex*

$$\Delta_\alpha = \{F \in \Delta(G, B) : \tau(F) \subseteq D(\alpha)^c\}.$$

gives rise to an exact sequence

$$0 \to \chi_q^\alpha \to 1_{P_\alpha}^G \xrightarrow{\partial} \bigoplus_{\beta \preccurlyeq_1 \alpha} 1_{P_\beta}^G \xrightarrow{\partial} \bigoplus_{\gamma \preccurlyeq_1 \beta} 1_{P_\gamma}^G \xrightarrow{\partial} \cdots \xrightarrow{\partial} 1_G^G \to 0 \tag{3.9}$$

where the boundary maps are given by

$$\partial : gP_\beta \mapsto \sum_{\gamma \preccurlyeq_1 \beta} [\beta : \gamma] \cdot gP_\gamma.$$

The *homology representation* χ_q^α is then defined as

$$\chi_q^\alpha := \ker\left(1_{P_\alpha}^G \to \bigoplus_{\beta \preccurlyeq_1 \alpha} 1_{P_\beta}^G\right) = \bigcap_{\beta \preccurlyeq_1 \alpha} \ker\left(1_{P_\alpha}^G \to 1_{P_\beta}^G\right).$$

The following decomposition of (left) G-modules is well-known (see e.g. Smith [52]):

$$1_B^G = \bigoplus_{\alpha \models r+1} \chi_q^\alpha. \tag{3.10}$$

On the other hand, Norton's decomposition of the 0-Hecke algebra $H_W(0)$ implies a decomposition of 1 into primitive orthogonal idempotents, i.e.

$$1 = \sum_{\alpha \models r+1} h_\alpha \overline{\pi}_{w_0(\alpha)} \pi_{w_0(\alpha^c)}, \quad h_\alpha \in H_W(0).$$

This decomposition of 1 into primitive orthogonal idempotents is explicitly given by Berg, Bergeron, Bhargava and Saliola [6], and is different from the one provided by Denton [17]. By the right action of $H_W(0)$ on 1_B^G, we have another decomposition of G-modules:

$$1_B^G = \bigoplus_{\alpha \models r+1} 1_B^G \, h_\alpha \overline{\pi}_{w_0(\alpha)} \pi_{w_0(\alpha^c)}. \tag{3.11}$$

Proposition 3.3.6. *The two G-module decompositions (3.10) and (3.11) are the same.*

Proof. Comparing (3.10) with (3.11) one sees that it suffices to show $1_B^G h_\alpha \overline{\pi}_{w_0(\alpha)} \pi_{w_0(\alpha^c)} \subseteq \chi_q^\alpha$. Assume

$$\overline{B} h_\alpha \overline{\pi}_{w_0(\alpha)} = \sum_i g_i \overline{B}, \quad g_i \in G.$$

For any $\beta \models r+1$ we have

$$\overline{B} \pi_{w_0(\beta^c)} = \overline{B W_{D(\beta)^c} B} = \overline{P}_\beta.$$

Hence

$$\overline{B} h_\alpha \overline{\pi}_{w_0(\alpha)} \pi_{w_0(\alpha^c)} = \sum_i g_i \overline{P}_\alpha \in 1_{P_\alpha}^G$$

and

$$\partial \left(\overline{B} h_\alpha \overline{\pi}_{w_0(\alpha)} \pi_{w_0(\alpha^c)} \right) = \sum_{\beta \preccurlyeq_1 \alpha} \pm \sum_i g_i \overline{P}_\beta = \sum_{\beta \preccurlyeq_1 \alpha} \pm \overline{B} h_\alpha \overline{\pi}_{w_0(\alpha)} \pi_{w_0(\beta^c)}.$$

If $\beta \preccurlyeq_1 \alpha$ then there exists $i \in D(\alpha) \cap D(\beta)^c$, and thus $\overline{\pi}_{w_0(\alpha)} \pi_{w_0(\beta^c)} = 0$. Therefore we are done. $\square$

One sees from (3.11) that χ_q^α is in general not a right $H_W(0)$-submodule of 1_B^G. However, when $G = GL(n, \mathbb{F}_q)$, one has that $\chi_q^{(n)}$ is the trivial representation and $\chi_q^{(1^n)}$ is the Steinberg representation of G [37], and both are right (isotypic) $H_W(0)$-modules.

3.3.3 Coinvariant algebra of (G, B)

In this subsection we study the action of the 0-Hecke algebra $H_n(0)$ on the coinvariant algebra $\mathbb{F}_q[X]^B/(\mathbb{F}_q[X]^G_+)$ of the pair (G, B), where $G = GL(n, \mathbb{F}_q)$ and B is the Borel subgroup of G.

Given a right $\mathbb{F}_qG$-module M, there is an isomorphism

$$\begin{array}{rcl} \mathrm{Hom}_{\mathbb{F}_qG}(1_B^G, M) & \xrightarrow{\sim} & M^B, \\ \phi & \mapsto & \phi(\overline{B}) \end{array}$$

with inverse map given by $\phi_m(\overline{B}) = m$ for all $m \in M^B$. The left $H_n(0)$-action $\overline{\pi}_w \overline{B} = \overline{BwB}$ on 1_B^G commutes with the right G-action and induces a left action on $\mathrm{Hom}_{\mathbb{F}_qG}(1_B^G, M)$ by

$$\overline{\pi}_w(\phi)(\overline{B}) = \phi(\overline{\pi}_{w^{-1}}\overline{B}) = \phi_m(\overline{Bw^{-1}B}).$$

Hence we have a left $H_n(0)$-action on M^B by

$$\overline{\pi}_w(m) = \overline{\pi}_w(\phi_m)(\overline{B}) = \phi_m(\overline{Bw^{-1}B}) = \phi_m(\overline{B}w^{-1}\overline{U}_{w^{-1}}) = mw^{-1}\overline{U}_{w^{-1}}.$$

The group G has a left action on $\mathbb{F}_q[X]$ by linear substitution, and this can be turned into a right action by $f \cdot g = g^{-1}f$ for all $f \in \mathbb{F}_q[X]$ and $g \in G$. Thus $H_n(0)$ has a left action on $\mathbb{F}_q[X]^B$ by

$$\overline{\pi}_w(f) = f \cdot w^{-1}\overline{U}_{w^{-1}} = \overline{U}_w w f, \quad \forall f \in \mathbb{F}_q[X]^B.$$

This action preserves the grading, and leaves the ideal $(\mathbb{F}_q[X]^G_+)$ stable: if $h_i \in \mathbb{F}_q[X]^G_+$, $f_i \in \mathbb{F}_q[X]^B$, then

$$\overline{\pi}_w\left(\sum_i h_i f_i\right) = \overline{U}_w w \left(\sum_i h_i f_i\right) = \sum_i h_i \overline{U}_w w(f_i).$$

Hence the coinvariant algebra $\mathbb{F}_q[X]^B/(\mathbb{F}_q[X]^G_+)$ of (G, B) becomes a graded $H_n(0)$-module.

Lemma 3.3.7. *If $Q = \mathbb{F}_q[X]^B/(\mathbb{F}_q[X]^G_+)$ and α is a composition of n, then*

$$Q_\alpha := \bigcap_{j \in D(\alpha)^c} \ker \overline{\pi}_j = \mathbb{F}_q[X]^{P_\alpha}/(\mathbb{F}_q[X]^G_+).$$

Proof. If $f \in \mathbb{F}_q[X]^{P_\alpha}$, then for all $j \notin D(\alpha)$ one has $U_{s_j}s_j \subseteq P_\alpha$ and hence

$$\overline{\pi}_j f = \overline{U}_{s_j} s_j f = |U_{s_j}| \cdot f = qf = 0.$$

Conversely, a B-invariant polynomial f gives rise to a P_α-invariant polynomial

$$\sum_{gB \in P_\alpha/B} gf = \sum_{w \in W_{D(\alpha)^c}} \overline{U}_w w f = \pi_{w_0(D(\alpha)^c)} f.$$

If $\overline{\pi}_j f$ belongs to the ideal $(\mathbb{F}_q[X]^G_+)$ for all $j \notin D(\alpha)$, so does $\overline{\pi}_w f \in (\mathbb{F}_q[X]^G_+)$ for all $w \in W_{D(\alpha)^c} \setminus \{1\}$. Thus $\pi_{w_0(\alpha^c)} f - f \in (\mathbb{F}_q[X]^G_+)$ and we are done. □

Theorem 3.3.8. *The $H_n(0)$-module $\mathbb{F}_q[X]^B/(\mathbb{F}_q[X]^G_+)$ has degree graded quasisymmetric characteristic*

$$\mathrm{Ch}_t\left(\mathbb{F}_q[X]^B/(\mathbb{F}_q[X]^G_+)\right) = \sum_{\alpha \models n} \begin{bmatrix} n \\ \alpha \end{bmatrix}_{q,t} M_\alpha = \sum_{\alpha \models n} r_\alpha(q,t) F_\alpha.$$

Proof. Let $Q = \mathbb{F}_q[X]^B/(\mathbb{F}_q[X]^G_+)$ and let $\alpha \models n$. It follows from Lemma 3.3.7 that

$$\mathrm{Hilb}(Q_\alpha, t) = \mathrm{Hilb}\left(\mathbb{F}_q[X]^{P_\alpha}/(\mathbb{F}_q[X]^G_+), t\right) = \begin{bmatrix} n \\ \alpha \end{bmatrix}_{q,t}.$$

Thus

$$c_\alpha(Q) = \sum_{\beta \preccurlyeq \alpha} (-1)^{\ell(\alpha)-\ell(\beta)} \begin{bmatrix} n \\ \beta \end{bmatrix}_{q,t} = r_\alpha(q,t).$$

Then the result follows immediately from Corollary 3.3.2. □

3.4 Cohomology ring of Springer fibers

In Section 3.1 we showed that the coinvariant algebra of $\mathfrak{S}_n$ is an $H_n(0)$-module whose graded quasisymmetric characteristic is the modified Hall-Littlewood symmetric function indexed by the partition 1^n. Now we generalize this result to partitions of hook shapes.

Throughout this section a partition of n is denoted by $\mu = (\mu_1, \ldots, \mu_n)$, where $0 \le \mu_1 \le \cdots \le \mu_n$. Denote $n(\mu) = \mu_{n-1} + 2\mu_{n-2} + \cdots + (n-1)\mu_1$. One can view

μ as a composition by dropping all the zero parts of μ. Then $n(\mu) = \mathrm{maj}(\mu)$, where $\mathrm{maj}(\alpha) = \sum_{i\in D(\alpha)} i$, for all compositions α.

Let V be an n-dimensional complex vector space. Fix a nilpotent matrix X_μ whose Jordan blocks are of size $\mu_1, \ldots, \mu_\ell$. The *Springer fiber* $\mathcal{F}_\mu$ corresponding to the partition μ is the variety of all flags $0 \subset V_1 \subset \cdots \subset V_n = V$ of subspaces $V_i \subseteq V$ satisfying $\dim V_i = i$ and $X_\mu(V_i) \subseteq V_{i-1}$. The cohomology ring of $\mathcal{F}_\mu$ is isomorphic to the ring $\mathbb{C}[X]/J_\mu$ for a certain homogeneous $\mathfrak{S}_n$-stable ideal J_μ, and carries an $\mathfrak{S}_n$-action that can be obtained from the $\mathfrak{S}_n$-action on $\mathbb{C}[X]$. In particular, if $\mu = 1^n$ then $\mathcal{F}_\mu$ is the flag variety G/B and $\mathbb{C}[X]/J_\mu$ is the coinvariant algebra of $\mathfrak{S}_n$.

Theorem 3.4.1 (Hotta-Springer [33], Garsia-Procesi [26])**.** *The graded Frobenius characteristic of $\mathbb{C}[X]/J_\mu$ is the modified Hall-Littlewood symmetric function*

$$\widetilde{H}_\mu(x;t) = \sum_\lambda t^{n(\mu)} K_{\lambda\mu}(t^{-1}) s_\lambda$$

where $K_{\lambda\mu}(t)$ is the Kostka-Foulkes polynomial.

To find an analogous result for the 0-Hecke algebras, we let $R_\mu := \mathbb{F}[X]/J_\mu$ where $\mathbb{F}$ is an arbitrary field, and consider the question of when the $H_n(0)$-action on $\mathbb{F}[X]$ preserves the ideal J_μ. Recall the following construction of J_μ by Tanisaki [58]. Let the conjugate of a partition μ of n be $\mu' = (0 \le \mu'_1 \le \cdots \le \mu'_n)$. Note that the *height* of the Young diagram of μ is $h = h(\mu) := \mu'_n$. Let

$$d_k(\mu) = \mu'_1 + \cdots + \mu'_k, \quad k = 1, \ldots, n.$$

Then the ideal J_μ is generated by

$$\{e_r(S) : k \ge r > k - d_k(\mu),\ |S| = k,\ S \subseteq \{x_1, \ldots, x_n\}\} \tag{3.12}$$

where $e_r(S)$ is the r-th elementary symmetric function in the set S of variables. See also Garsia and Procesi [26].

Proposition 3.4.2. *The Demazure operators preserve the ideal J_μ if and only if μ is a hook.*

Example 3.4.3. We give some examples before proving this result.

First let $\mu = 114000$ be a partition of $n = 6$, which is a hook of height $h = 3$. Since $\mu' = 001113$, one has

$$(d_k(\mu) : 1 \leq k \leq 6) = (0, 0, 1, 2, 3, 6),$$

$$(k - d_k(\mu) : 1 \leq k \leq 6) = (1, 2, 2, 2, 2, 0).$$

Thus J_μ is generated by $e_i = e_i(x_1, \ldots, x_6)$ for $i = 1, \ldots, 6$, and

$$\{e_3(S) : |S| = 3\} \cup \{e_3(S),\ e_4(S) : |S| = 4\} \cup \{e_3(S),\ e_4(S),\ e_5(S) : |S| = 5\}. \quad (3.13)$$

The first set appearing in the above union (3.13) can be written as

$$\mathcal{M}_3 := \{x_i x_j x_k : 1 \leq i < j < k \leq 6\};$$

the second and third sets in (3.13) are redundant for J_μ, as their elements belong to the ideal generated by $\mathcal{M}_3$. Therefore J_μ is generated by $\{e_1, \ldots, e_6\} \cup \mathcal{M}_3$ (actually e_4, e_5, and e_6 are also redundant). We already know that the Demazure operators are linear over $e_1, \ldots, e_6$; if $1 \leq i_1 < i_2 < i_3 \leq 6$ and f is an arbitrary monomial then it follows from (3.3) that $\overline{\pi}_i(x_{i_1} x_{i_2} x_{i_3} f)$ is divisible by some element in $\mathcal{M}_3$. Thus the ideal J_μ is $H_n(0)$-stable.

Now we look at the partition $\mu = (2, 2)$ of $n = 4$, which is not a hook. One has $h = 2$, $\mu' = \mu = 0022$, and $(k - d_k(\mu')) = (1, 2, 1, 0)$. Thus J_μ is generated by

$$\{e_2(S),\ e_3(S) : |S| = 3\} \cup \{e_1,\ e_2,\ e_3,\ e_4\}.$$

If J_μ is $H_n(0)$-stable, then $\pi_3(e_2(x_1, x_2, x_3)) = x_1x_4 + x_2x_4 \in J_\mu$ and thus

$$e_2(x_1, x_2, x_4) - \pi_3(e_2(x_1, x_2, x_3)) = x_1x_2 \in J_\mu.$$

But one can check $x_1x_2 \notin J_\mu$. This contradiction shows that $J_{(2,2)}$ is not $H_4(0)$-stable.

Proof. The proof is similar to the above example. We first assume μ is a hook, i.e. $\mu = (0^{n-h}, 1^{h-1}, n - h + 1)$. Then $\mu' = (0^{h-1}, 1^{n-h}, h)$ and so

$$(1 - d_1(\mu), 2 - d_2(\mu), \ldots, n - d_n(\mu)) = (1, 2, \ldots, h - 1, h - 1, \ldots, h - 1, 0).$$

It follows that the ideal J_μ is generated by the elementary symmetric functions $e_1, \ldots, e_n$, together with the following partial elementary symmetric functions

$$\{e_r(S) : r = h, \ldots, k,\ S \subseteq \{x_1, \ldots, x_n\},\ |S| = k,\ k = h, \ldots, n - 1\}.$$

These partial elementary symmetric functions all belong to the ideal generated by

$$\mathcal{M}_h = \{x_{i_1} \cdots x_{i_h} : 0 \leq i_1 < \cdots < i_h \leq n\}.$$

Thus J_μ is generated by $\{e_1, \ldots, e_n\} \cup \mathcal{M}_h$.

We know that the Demazure operators are e_i-linear for all $i \in [n]$. By (3.3), if $x_{i_1} \cdots x_{i_h}$ is in $\mathcal{M}_h$ and f is an arbitrary monomial, then $\overline{\pi}_i(x_{i_1} \cdots x_{i_h} f)$ is divisible by some element in $\mathcal{M}_h$. Thus the ideal J_μ is preserved by the Demazure operators.

Now assume μ is not a hook. Then $\mu'_{n-1} \geq 2$ and thus

$$\begin{aligned} k - d_k(\mu) &= k - n + n - d_k(\mu) \\ &= k - n + \mu'_n + \mu'_{n-1} + \cdots + \mu'_{k+1} \\ &\geq k - n + \mu'_n + 2 + 1 + \cdots + 1 \\ &= \mu'_n = h \end{aligned}$$

for $k = n-2, \ldots, n - \mu_1 + 1$. One also sees that

$$k - d_k(\mu) = \begin{cases} 0, & k = n, \\ h-1, & k = n-1, \\ k, & n - \mu_1 \geq k \geq 1. \end{cases}$$

Thus the only elements in the generating set (3.12) that have degree no more than h are $e_1, \ldots, e_h$ and those $e_h(S)$ with $|S| = n-1$.

Suppose to the contrary that J_μ is preserved by Demazure operators. Since

$$e_h(x_1, \ldots, x_{n-1}) = x_{n-1} e_{h-1}(x_1, \ldots, x_{n-2}) + e_h(x_1, \ldots, x_{n-2}) \in J_\mu$$

we have

$$\overline{\pi}_{n-1} e_h(x_1, \ldots, x_{n-1}) = x_n e_{h-1}(x_1, \ldots, x_{n-2}) \in J_\mu$$

and thus

$$e_h(x_1, \ldots, x_{n-2}, x_n) - \overline{\pi}_{n-1} e_h(x_1, \ldots, x_{n-1}) = e_h(x_1, \ldots, x_{n-2}) \in J_\mu.$$

Repeating this process one obtains $e_h(x_1, \ldots, x_h) = x_1 \cdots x_h \in J_\mu$. Then applying the Demazure operators to $x_1 \cdots x_h$ gives $x_{i_1} \cdots x_{i_h} \in J_\mu$ whenever $1 \leq i_1 < \cdots < i_h \leq n$. Considering the degree we have

$$x_{i_1} \cdots x_{i_h} = \sum_{i=1}^{h} f_i e_i + \sum_{|S|=n-1} c_S e_h(S)$$

where $f_i \in \mathbb{F}[X]$ is homogeneous of degree $h-i$ and $c_S \in \mathbb{F}$. Descending this equation to the coinvariant algebra $\mathbb{F}[X]/(e_1, \ldots, e_n)$, one obtains an expression of an arbitrary element in

$$\mathcal{M}_{h,n-1} := \{x_{i_1} \cdots x_{i_h} : 1 \le i_1 < \cdots < i_h \le n-1\}$$

as an $\mathbb{F}$-linear combination of $\{e_h(S) : |S| = n-1\}$. Thus the $\mathbb{F}$-subspaces U and U' of $\mathbb{F}[X]/(e_1, \ldots, e_n)$ spanned respectively by $\mathcal{M}_{h,n-1}$ and $\{e_h(S) : |S| = n-1\}$ satisfy the relation $U \subseteq U'$.

It is well-known that all divisors of $x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$ form an $\mathbb{F}$-basis for the coinvariant algebra $\mathbb{F}[X]/(e_1, \ldots, e_n)$, i.e. the *Artin basis* [4]. Thus

$$n = \binom{n}{n-1} \ge \dim U' \ge \dim U = \binom{n-1}{h} \ge \binom{n-1}{2}$$

where the last inequality follows from $2 \le h \le n-2$ since μ is not a hook. Therefore we must have $n = 4$, $h = 2$, and $\mu = (2,2)$. But we know from the example preceding this proof that $J_{(2,2)}$ is not $H_4(0)$-stable. Thus the proof is complete. $\square$

Theorem 3.4.4. *Assume $\mu = (0^{n-h}, 1^{h-1}, n-h+1)$ is a hook and view it as a composition by removing all zeros. Then the $H_n(0)$-module R_μ is a direct sum of the projective indecomposable $H_n(0)$-modules $\mathbf{P}_\alpha$ for all compositions $\alpha \preccurlyeq \mu$, i.e.*

$$R_\mu \cong \bigoplus_{\alpha \preccurlyeq \mu} \mathbf{P}_\alpha.$$

Proof. By the proof of the previous proposition, J_μ is generated by $e_1, \ldots, e_n$ and

$$\mathcal{M}_h = \{x_{i_1} \cdots x_{i_h} : 1 \le i_1 < \cdots < i_h \le n\}.$$

Thus R_μ is the quotient of $\mathbb{F}[X]/(\mathbb{F}[X]_+^{\mathfrak{S}_n})$ by its ideal generated by $\mathcal{M}_h$. By Theorem 3.1.6, it suffices to show that $\mathbb{F}[X]/J_\mu$ has a basis given by

$$\{\overline{\pi}_w x_{D(w)} : w \in \mathfrak{S}_n, D(w) \subseteq D(\mu)\}. \tag{3.14}$$

If $D(w) \not\subseteq D(\mu) = \{1, 2, \ldots, h-1\}$, i.e. w has a descent $i \ge h$, then $x_{D(w)}$ contains at least h distinct variables, and so do all monomials in $\overline{\pi}_w x_{D(w)}$ by (3.3). Thus $\mathbb{F}[X]/J_\mu$ is spanned by (3.14).

To show (3.14) is linearly independent, assume

$$\sum_{D(w)\subseteq D(\mu)} c_w \overline{\pi}_w x_{D(w)} \in J_\mu, \quad c_w \in \mathbb{F}.$$

For any polynomial $f \in \mathbb{F}[X]$, let $[f]_h$ be the polynomial obtained from f by removing all terms divisible by some element in $\mathcal{M}_h$. It follows that

$$\sum_{D(w)\subseteq D(\mu)} c_w [\overline{\pi}_w x_{D(w)}]_h \in (e_1, \ldots, e_{h-1}).$$

If $D(w) \subseteq D(\mu) = [h-1]$ then the leading term of $[\overline{\pi}_w x_{D(w)}]_h$ under "$\prec$" is still the descent monomial $wx_{D(w)}$. By Lemma 3.1.5,

$$\left\{ [\overline{\pi}_w x_{D(w)}]_h : w \in \mathfrak{S}_n, D(w) \subseteq D(\mu) \right\}$$

gives a linearly independent set in $\mathbb{F}[X]/(\mathbb{F}[X]_+^{\mathfrak{S}_n})$. It follows that $c_w = 0$ whenever $D(w) \subseteq D(\mu)$. $\square$

By work of Bergeron and Zabrocki [8], the modified Hall-Littlewood functions $\widetilde{H}_\mu(x;t)$ have the following noncommutative analogue lying in $\mathbf{NSym}[t]$ for all compositions α:

$$\widetilde{\mathbf{H}}_\alpha(\mathbf{x};t) := \sum_{\beta \preccurlyeq \alpha} t^{\mathrm{maj}(\beta)} \mathbf{s}_\beta.$$

Corollary 3.4.5. *Assume μ is a hook. Then*

$$\mathbf{ch}_t(R_\mu) = \sum_{\alpha \preccurlyeq \mu} t^{\mathrm{maj}(\alpha)} \mathbf{s}_\alpha = \widetilde{\mathbf{H}}_\mu(\mathbf{x};t), \tag{3.15}$$

$$\mathrm{Ch}_t(R_\mu) = \sum_{\alpha \preccurlyeq \mu} t^{\mathrm{maj}(\alpha)} s_\alpha = \widetilde{H}_\mu(x;t). \tag{3.16}$$

Proof. Theorem 3.4.4 immediately implies (3.15). The degree graded quasisymmetric characteristic of R_μ is the commutative image of $\mathbf{ch}_t(R_\mu)$, which equals $\widetilde{H}_\mu(x;t)$ when μ is a hook, according to Bergeron and Zabrocki [8, Proposition 9]. $\square$

Remark 3.4.6. (i) We showed in Section 3.1 that the coinvariant algebra $\mathbb{F}[X]/(\mathbb{F}[X]_+^{\mathfrak{S}_n})$ carries the regular representation of $H_n(0)$; one also sees this from (3.15) with $\mu = 1^n$.

(ii) Using certain difference operators, Hivert [29] defined a noncommutative analogue of the Hall-Littlewood functions, which is in general different from the noncommutative analogue of Bergeron and Zabrocki. However, they are the same when μ is a hook!
(iii) It is not clear to the author why the results are nice only in the hook case, except for a naive explanation: the hooks are the only diagrams that belong to both the family of the Young diagrams of partitions and the family of ribbon diagrams of compositions.

3.5 Questions for future research

3.5.1 Equidistribution of the inversion number and major index

The equidistribution of inv and maj was first proved on permutations of multisets by P.A. MacMahon in the 1910s; applying an inclusion-exclusion would give their equidistribution on inverse descent classes of $\mathfrak{S}_n$. However, the first proof for the latter result appearing in the literature was by Foata and Schützenberger [23] in 1970, using a bijection constructed earlier by Foata [22]. Is there an algebraic proof from the (q,t)-bigraded characteristic of $\mathbb{F}[\mathbf{x}]/(\mathbb{F}[\mathbf{x}]_+^{\mathfrak{S}_n})$, which is given in Corollary 3.1.10 (i) and involves inv, maj, and inverse descents?

3.5.2 Decompositions of 1_B^G and $\mathbb{F}[\mathbf{x}]^B/(\mathbb{F}[\mathbf{x}]_+^G)$

In § 3.3 we studied an $H_W(0)$-action on the flag variety 1_B^G and found its simple composition factors, but we do *not* know the decomposition of 1_B^G into indecomposable $H_W(0)$-modules. Assume $G = GL(n, \mathbb{F}_q)$ below. Computations show that 1_B^G is in general *not* projective, although its quasisymmetric characteristic is always symmetric.

The coinvariant algebra $\mathbb{F}[\mathbf{x}]^B/(\mathbb{F}[\mathbf{x}]_+^G)$ is *not* a projective $H_n(0)$-module either, since its graded quasisymmetric characteristic is *not* even symmetric (see the definition of the (q,t)-multinomial coefficients). To find its decomposition, it will be helpful to know more (nonprojective) indecomposable $H_n(0)$-modules (there are infinitely many, and some were studied by Duchamp, Hivert, and Thibon [19]).

Another question is to find a q-analogue of the Demazure operators, which might give another $H_n(0)$-action on $\mathbb{F}[\mathbf{x}]^B/(\mathbb{F}[\mathbf{x}]_+^G)$.

3.5.3 Coincidence of Frobenius type characteristics

For $G = GL(n, \mathbb{C})$, the complex flag variety 1_B^G has its cohomology ring isomorphic to the coinvariant algebra of $\mathfrak{S}_n$, whose graded Frobenius characteristic and graded quasisymmetric characteristic both equal the modified Hall-Littlewood symmetric function $\widetilde{H}_{1^n}(x;t)$. For $G = GL(n, \mathbb{F}_q)$, the flag variety 1_B^G itself, when defined over a field of characteristic $p \mid q$, is also an $H_n(0)$-module whose quasisymmetric characteristic equals $\widetilde{H}_{1^n}(x;q)$. Is there a better explanation for the coincidence of these Frobenius type characteristics?

Chapter 4

0-Hecke algebra action on the Stanley-Reisner ring of the Boolean algebra

In the previous chapter we studied the $H_n(0)$-action on the polynomial ring $\mathbb{F}[X]$. In this chapter we define an action of $H_n(0)$ on the Stanley-Reisner ring of the Boolean algebra, which turns out to be a natural analogue of the polynomial ring $\mathbb{F}[X]$, with nice behavior under the $H_n(0)$-action.

4.1 Stanley-Reisner ring of the Boolean algebra

In this section we study the Stanley-Reisner ring of the Boolean algebra.

4.1.1 Boolean algebra

The Boolean algebra $\mathcal{B}_n$ is the ranked poset of all subsets of $[n] := \{1, 2, \ldots, n\}$ ordered by inclusion, with minimum element $\emptyset$ and maximum element $[n]$. The rank of a subset of $[n]$ is defined as its cardinality. The Stanley-Reisner ring $\mathbb{F}[\mathcal{B}_n]$ of the Boolean algebra $\mathcal{B}_n$ is the quotient of the polynomial algebra $\mathbb{F}[y_A : A \subseteq [n]]$ by the ideal

$$(y_A y_B : \ A \text{ and } B \text{ are incomparable in } \mathcal{B}_n)\,.$$

It has an $\mathbb{F}$-basis $\{y_M\}$ indexed by the multichains M in $\mathcal{B}_n$, and is multigraded by the rank multisets $r(M)$ of the multichains M.

The symmetric group $\mathfrak{S}_n$ acts on the Boolean algebra $\mathcal{B}_n$ by permuting the integers $1,\ldots,n$. This induces an $\mathfrak{S}_n$-action on the Stanley-Reisner ring $\mathbb{F}[\mathcal{B}_n]$, preserving its multigrading. The *invariant algebra* $\mathbb{F}[\mathcal{B}_n]^{\mathfrak{S}_n}$ consists of all elements in $\mathbb{F}[\mathcal{B}_n]$ invariant under this $\mathfrak{S}_n$-action. For $i=0,1,\ldots,n$, the rank polynomial $\theta_i := \sum_{|A|=i} y_A$ is obviously invariant under the $\mathfrak{S}_n$-action; the converse is also true.

Proposition 4.1.1. *The invariant algebra* $\mathbb{F}[\mathcal{B}_n]^{\mathfrak{S}_n}$ *equals* $\mathbb{F}[\Theta]$*, where* $\Theta := \{\theta_0,\ldots,\theta_n\}$*.*

Proof. It suffices to show $\mathbb{F}[\mathcal{B}_n]^{\mathfrak{S}_n} \subseteq \mathbb{F}[\Theta]$. The $\mathfrak{S}_n$-action on $\mathbb{F}[\mathcal{B}_n]$ breaks up the set of nonzero monomials into orbits, and the orbit sums form an $\mathbb{F}$-basis for $\mathbb{F}[\mathcal{B}_n]^{\mathfrak{S}_n}$. The $\mathfrak{S}_n$-orbit of a nonzero monomial with rank multiset $\{0^{a_0},\ldots,n^{a_n}\}$ consists of all nonzero monomials with the same rank multiset, and the corresponding orbit sum equals $\theta_0^{a_0}\cdots\theta_n^{a_n}$. This completes the proof. $\square$

Garsia [24] showed that $\mathbb{F}[\mathcal{B}_n]$ is a free $\mathbb{F}[\Theta]$-module on the basis of descent monomials

$$Y_w := \prod_{i\in D(w)} y_{\{w(1),\ldots,w(i)\}}, \quad \forall w\in\mathfrak{S}_n.$$

Example 4.1.2. Let $n=3$. The Boolean algebra $\mathcal{B}_3$ consists of all subsets of $\{1,2,3\}$. Its Stanley-Reisner ring $\mathbb{F}[\mathcal{B}_3]$ is a free $\mathbb{F}[\Theta]$-module with a basis of descent monomials, where Θ consists of the rank polynomials

$$\theta_0 = y_\emptyset,\ \theta_1 := y_1+y_2+y_3,\ \theta_2 := y_{12}+y_{13}+y_{23},\ \theta_3 := y_{123}$$

and the descent monomials are

$$Y_1 := 1,\ Y_{s_1} := y_2,\ Y_{s_2s_1} := y_3,\ Y_{s_2} := y_{13},\ Y_{s_1s_2} := y_{23},\ Y_{s_1s_2s_1} := y_{23}y_3.$$

4.1.2 Multichains in $\mathcal{B}_n$

We study the multichains in $\mathcal{B}_n$, as they naturally index an $\mathbb{F}$-basis for $\mathbb{F}[\mathcal{B}_n]$. We first recall some notation from Chapter 1. A *weak composition* with length $k\ge 0$ is a sequence $\alpha=(\alpha_1,\ldots,\alpha_k)$ of k *nonnegative* (positive for a composition) integers. The *size* of α is $|\alpha| := \alpha_1+\cdots+\alpha_n$. If $|\alpha|=n$ then we say α is a weak composition of

n. We denote by $\mathrm{Com}(n,k)$ the set of all weak compositions of n with length k. The *descent multiset* of α is the *multiset* $D(\alpha) := \{\alpha_1, \alpha_1+\alpha_2, \alpha_1+\cdots+\alpha_{k-1}\}$. The map $\alpha \mapsto D(\alpha)$ gives a bijection between weak compositions of n and multisets with elements in $\{0,\ldots,n\}$. The parabolic subgroup $\mathfrak{S}_\alpha$ is the same as the parabolic subgroup of $\mathfrak{S}_n$ indexed by the underlying composition of α obtained by removing all zeros from α; similarly for $\mathfrak{S}^\alpha$.

The homogeneous components of $\mathbb{F}[\mathcal{B}_n]$ are indexed by multisets with elements in $\{0,\ldots,n\}$, or equivalently by weak compositions α of n. The α-homogeneous component $\mathbb{F}[\mathcal{B}_n]_\alpha$ has an $\mathbb{F}$-basis $\{y_M : r(M) = D(\alpha)\}$.

Let $M = (A_1 \subseteq \cdots \subseteq A_k)$ be an arbitrary multichain of length k in $\mathcal{B}_n$; set $A_0 := \emptyset$ and $A_{k+1} := [n]$ by convention. Define $\alpha(M) := (\alpha_1,\ldots,\alpha_{k+1})$, where $\alpha_i = |A_i| - |A_{i-1}|$ for all $i \in [k+1]$. Then $\alpha(M) \in \mathrm{Com}(n,k+1)$ and $D(\alpha(M)) = r(M)$, i.e. $\alpha(M)$ indexes the homogeneous component containing y_M. Define $\sigma(M)$ to be the minimal element in $\mathfrak{S}_n$ which sends the standard multichain $[\alpha_1] \subseteq [\alpha_1+\alpha_2] \subseteq \cdots \subseteq [\alpha_1+\cdots+\alpha_k]$ with rank multiset $D(\alpha(M))$ to M. Then $\sigma(M) \in \mathfrak{S}^{\alpha(M)}$.

The map $M \mapsto (\alpha(M), \sigma(M))$ is a bijection between multichains of length k in $\mathcal{B}_n$ and the pairs (α,σ) of $\alpha \in \mathrm{Com}(n,k+1)$ and $\sigma \in \mathfrak{S}^\alpha$. A short way to write down this encoding of M is to insert bars at the descent positions of $\sigma(M)$. For example, the length-4 multichain $\{2\} \subseteq \{2\} \subseteq \{1,2,4\} \subseteq [4]$ in $\mathcal{B}_4$ is encoded by $2||14|3|$.

There is another way to encode the multichain M. Let $p(M) := (p_1(M),\ldots,p_n(M))$, where $p_i(M)$ is the first position where i appears in M, i.e.

$$p_i(M) := \min\{j \in [k+1] : i \in A_j\},$$

for $i = 1,\ldots,n$. One checks that

$$\begin{cases} p_i(M) > p_{i+1}(M) \Leftrightarrow i \in D(\sigma(M)^{-1}), \\ p_i(M) = p_{i+1}(M) \Leftrightarrow i \notin D(\sigma(M)^{-1}),\ D(s_i\sigma(M)) \not\subseteq D(\alpha(M)), \\ p_i(M) < p_{i+1}(M) \Leftrightarrow i \notin D(\sigma(M)^{-1}),\ D(s_i\sigma(M)) \subseteq D(\alpha(M)). \end{cases} \tag{4.1}$$

This will be used later when we study the $H_n(0)$-action on $\mathbb{F}[\mathcal{B}_n]$. The map $M \mapsto p(M)$ is an bijection between the set of multichains with length k in $\mathcal{B}_n$ and the set $[k+1]^n$ of all words of length n on the alphabet $[k+1]$, for any fixed integer $k \geq 0$.

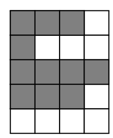
Suppose that $p(M) = \mathbf{p} = (p_1, \ldots, p_n) \in [k+1]^n$. We define

$$\mathrm{inv}(\mathbf{p}) := \#\{(i,j) : 1 \leq i < j \leq n,\ p_i > p_j\}.$$

One sees that $\mathrm{inv}(\sigma(M)) = \mathrm{inv}(p(M))$. Let $\mathbf{p}' := (p'_1, \ldots, p'_k)$ where

$$p'_i := |\{j : p_j(M) \leq i\}| = |A_i|.$$

Then the rank multiset of M consists of $p'_1, \ldots, p'_k$. If we draw $k+1-p_j$ boxes on the j-th row of a $n \times k$ rectangle for all $j \in [n]$, then p'_i is the number of boxes on the $(k+1-i)$-th column. For example, the multichain $3|14||2|5$ corresponds to $\mathbf{p} = (2,4,1,2,5) \in [5]^5$, and one has $\mathbf{p}' = (1,3,3,4)$ and the following picture.



This implies an equation which will be used later:

$$(q^k + \cdots + q + 1)^n = \sum_{\mathbf{p} \in [k+1]^n} \prod_{1 \leq j \leq n} q^{k+1-p_j} = \sum_{\mathbf{p} \in [k+1]^n} \prod_{1 \leq i \leq k} q^{p'_i}. \tag{4.2}$$

Define $D(\mathbf{p}) := \{i \in [n-1] : p_i > p_{i+1}\}$. For example, $D(2,5,1,2,4) = \{2\}$.

These two encodings $(\alpha(M), \sigma(M))$ and $p(M)$ of the multichains M in the Boolean algebra $\mathcal{B}_n$ were already used by Garsia and Gessel [25] in their work on generating functions of multivariate distributions of permutation statistics (with slightly different notation). In the next section we will use these encodings to derive multivariate quasisymmetric function identities from $H_n(0)$-action on the Stanley-Reisner ring of $\mathcal{B}_n$, giving generalizations of some results of Garsia and Gessel [25].

4.1.3 Rank-selection

Let α be a composition of n. We define the *rank-selected Boolean algebra*

$$\mathcal{B}_\alpha := \{A \subseteq [n] : |A| \in D(\alpha) \cup \{n\}\}$$

which is a ranked subposet of the Boolean algebra $\mathcal{B}_n$. We always exclude $\emptyset$ but keep $[n]$ because there is a nice analogy between the Stanley-Reisner ring of $\mathcal{B}_n^* := \mathcal{B}_{1^n} = \mathcal{B}_n \setminus \{\emptyset\}$ and the polynomial ring $\mathbb{F}[X]$.

To explain this analogy, we use the *transfer map* $\tau : \mathbb{F}[\mathcal{B}_n] \to \mathbb{F}[X]$ defined by

$$\tau(y_M) := \prod_{1 \le i \le k} \prod_{j \in A_i} x_j$$

for all multichains $M = (A_1 \subseteq \cdots \subseteq A_k)$ in $\mathcal{B}_n$. This transfer map is *not* a ring homomorphism (e.g. $y_{\{1\}} y_{\{2\}} = 0$ but $x_1 x_2 \ne 0$). Nevertheless, it restricts to an isomorphism $\tau : \mathbb{F}[\mathcal{B}_n^*] \cong \mathbb{F}[X]$ of $\mathfrak{S}_n$-modules.

Proposition 4.1.1 showed that the invariant algebra $\mathbb{F}[\mathcal{B}_n^*]^{\mathfrak{S}_n}$ equals the polynomial algebra $\mathbb{F}[\theta_1, \ldots, \theta_n]$, and as mentioned in §4.1.1, $\mathbb{F}[\mathcal{B}_n^*]$ is a free $\mathbb{F}[\theta_1, \ldots, \theta_n]$-module on the descent basis $\{Y_w : w \in \mathfrak{S}_n\}$. The transfer map τ sends the rank polynomials $\theta_1, \ldots, \theta_n$ to the elementary symmetric polynomials $e_1, \ldots, e_n$, which generate the invariant algebra $\mathbb{F}[X]^{\mathfrak{S}_n}$. It also sends the descent monomials Y_w in $\mathbb{F}[\mathcal{B}_n^*]$ to the descent monomials

$$X_w := \prod_{i \in D(w)} x_{w(1)} \cdots x_{w(i)}$$

in $\mathbb{F}[X]$ for all $w \in \mathfrak{S}_n$, which form a free $\mathbb{F}[X]^{\mathfrak{S}_n}$-basis for $\mathbb{F}[X]$ (see e.g. Garsia [24]). Therefore $\mathbb{F}[\mathcal{B}_n^*]$ is in a nice analogy with $\mathbb{F}[X]$ via the transfer map τ.

Remark 4.1.3. The Stanley-Reisner ring $\mathbb{F}[\mathcal{B}_n]$ is not much different from $\mathbb{F}[\mathcal{B}_n^*]$, as one can see the $\mathbb{F}$-algebra isomorphisms $\mathbb{F}[\mathcal{B}_n] \cong \mathbb{F}[\mathcal{B}_n^*] \otimes_{\mathbb{F}} \mathbb{F}[\theta_0]$ and $\mathbb{F}[\mathcal{B}_n]/(\theta_0) \cong \mathbb{F}[\mathcal{B}_n^*]$, where $\theta_0 = y_\emptyset$.

In general, the Stanley-Reisner ring of the rank-selected Boolean algebra $\mathcal{B}_\alpha$ is the multigraded subalgebra of $\mathbb{F}[\mathcal{B}_n]$ generated by $\{y_A : |A| \in D(\alpha) \cup \{n\}\}$. There is also a projection $\phi_\alpha : \mathbb{F}[\mathcal{B}_n] \twoheadrightarrow \mathbb{F}[\mathcal{B}_\alpha]$ of multigraded algebras given by

$$\phi_\alpha(y_A) := \begin{cases} y_A, & \text{if } A \subseteq [n],\ |A| \in D(\alpha) \cup \{n\}, \\ 0, & \text{if } A \subseteq [n],\ |A| \notin D(\alpha) \cup \{n\}. \end{cases}$$

The $\mathfrak{S}_n$-action preserves both the inclusion $\mathbb{F}[\mathcal{B}_\alpha] \subseteq \mathbb{F}[\mathcal{B}_n]$ and the projection ϕ_α. Thus one has an isomorphism

$$\mathbb{F}[\mathcal{B}_n]/(A \subseteq [n] : |A| \notin D(\alpha) \cup \{n\}) \cong \mathbb{F}[\mathcal{B}_\alpha]$$

of multigraded $\mathbb{F}$-algebras and $\mathfrak{S}_n$-modules.

Applying the projection ϕ_α one sees that the invariant algebra $\mathbb{F}[\mathcal{B}_\alpha]^{\mathfrak{S}_n}$ is the polynomial algebra $\mathbb{F}[\Theta_\alpha]$, where $\Theta_\alpha := \{\theta_i : i \in D(\alpha) \cup \{n\}\}$, and $\mathbb{F}[\mathcal{B}_\alpha]$ is a free $\mathbb{F}[\Theta_\alpha]$-module on the basis of descent monomials Y_w for all $w \in \mathfrak{S}^\alpha$.

4.2 0-Hecke algebra action

In this section we define an action of the 0-Hecke algebra $H_n(0)$ on the Stanley-Reisner ring $[\mathcal{B}_n]$ and establish Theorem 1.3.1 and Theorem 1.3.2.

4.2.1 Definition

We saw an analogy between $\mathbb{F}[\mathcal{B}_n]$ and $\mathbb{F}[X]$ in the last section. The usual $H_n(0)$-action on the polynomial ring $\mathbb{F}[X]$ is via the *Demazure operators*

$$\overline{\pi}_i(f) := \frac{x_{i+1}f - x_{i+1}s_i f}{x_i - x_{i+1}}, \quad \forall f \in \mathbb{F}[X],\ 1 \le i \le n-1. \tag{4.3}$$

The above definition is equivalent to

$$\overline{\pi}_i(x_i^a x_{i+1}^b m) = \begin{cases} (x_i^{a-1}x_{i+1}^{b+1} + x_i^{a-2}x_{i+1}^{b+2} \cdots + \underline{x_i^b x_{i+1}^a})m, & \text{if } a > b, \\ 0, & \text{if } a = b, \\ -(\underline{x_i^a x_{i+1}^b} + x_i^{a+1}x_{i+1}^{b-1} + \cdots + x_i^{b-1}x_{i+1}^{a+1})m, & \text{if } a < b. \end{cases} \tag{4.4}$$

Here m is a monomial in $\mathbb{F}[X]$ containing neither x_i nor x_{i+1}. Denote by $\overline{\pi}'_i$ the operator obtained from (4.4) by taking only the leading term (underlined) in the lexicographic order of the result. Then $\overline{\pi}'_1, \ldots, \overline{\pi}'_{n-1}$ realize another $H_n(0)$-action on $\mathbb{F}[X]$. We call it the *transferred* $H_n(0)$*-action* because it can be obtained by applying the transfer map τ to our $H_n(0)$-action on $\mathbb{F}[\mathcal{B}_n]$, which we now define.

Let $M = (A_1 \subseteq \cdots \subseteq A_k)$ be a multichain in $\mathcal{B}_n$. Recall that

$$p_i(M) := \min\{j \in [k+1] : i \in A_j\}$$

for all $i \in [n]$. We define

$$\overline{\pi}_i(y_M) := \begin{cases} -y_M, & p_i(M) > p_{i+1}(M), \\ 0, & p_i(M) = p_{i+1}(M), \\ s_i(y_M), & p_i(M) < p_{i+1}(M) \end{cases} \tag{4.5}$$

for $i = 1, \ldots, n-1$. Applying the transfer map τ one recovers $\overline{\pi}'_i$. For instance, when $n = 4$ one has

$$\overline{\pi}_1(y_{1|34||2|}) = y_{2|34||1|}, \quad \overline{\pi}_2(y_{1|34||2|}) = -y_{1|34||2|}, \quad \overline{\pi}_3(y_{1|34||2|}) = 0.$$

Applying the transfer map one has

$$\overline{\pi}_1'(x_1^4x_2x_3^3x_4^3) = x_1x_2^4x_3^3x_4^3, \quad \overline{\pi}_2'(x_1^4x_2x_3^3x_4^3) = -x_1^4x_2x_3^3x_4^3, \quad \overline{\pi}_3'(x_1^4x_2x_3^3x_4^3) = 0.$$

It is easy to check the relations $\overline{\pi}_i^2 = -\overline{\pi}_i$ and $\overline{\pi}_i\overline{\pi}_j = \overline{\pi}_j\overline{\pi}_i$ whenever $1 \le i, j \le n-1$ and $|i-j| > 1$. For any $i \in [n-2]$, by considering different possibilities for the relative positions of $p_i(M)$, $p_{i+1}(M)$, and $p_{i+2}(M)$, one verifies $\overline{\pi}_i\overline{\pi}_{i+1}\overline{\pi}_i = \overline{\pi}_{i+1}\overline{\pi}_i\overline{\pi}_{i+1}$ case by case. Hence $H_n(0)$ acts on $\mathbb{F}[\mathcal{B}_n]$ via the above defined operators $\overline{\pi}_1, \ldots, \overline{\pi}_{n-1}$. This $H_n(0)$-action preserves the multigrading of $\mathbb{F}[\mathcal{B}_n]$, and thus restricts to the Stanley-Reisner ring $\mathbb{F}[\mathcal{B}_\alpha]$ for any composition α of n.

Another way to see that $\overline{\pi}_1, \ldots, \overline{\pi}_{n-1}$ realize an $H_n(0)$-action on $\mathbb{F}[\mathcal{B}_n]$ preserving the multigrading is to show that each homogeneous component of $\mathbb{F}[\mathcal{B}_n]$ is isomorphic to an $H_n(0)$-module. We will give this in Lemma 4.2.13.

Remark 4.2.1. Our $H_n(0)$ action on $\mathbb{F}[\mathcal{B}_n]$ also has a similar expression to the Demazure operator (4.3) as one can show that it has the following properties.

(i) If f, g, h are elements in $\mathbb{F}[\mathcal{B}_n]$ such that $f = gh$ and h is homogeneous, then there exists a unique element g', defined as the quotient f/h, such that $f = g'h$ and $y_M h \neq 0$ for every monomial y_M appearing in g'.

(ii) Suppose that $i \in [n-1]$ and $M = (A_1 \subseteq \cdots \subseteq A_k)$ is a multichain in $\mathcal{B}_n$. Let j be the largest integer in $\{0, \ldots, k\}$ such that $A_j \cap \{i, i+1\} = \emptyset$. Then $s_i(y_M) = y_{M^{(i)}} s_i(y_{M_i})$ and $\overline{\pi}_i(y_M) = y_{M^{(i)}}\overline{\pi}_i(y_{M_i})$, where $M^{(i)} := (A_1 \subseteq \cdots \subseteq A_j)$ and $M_i := (A_{j+1} \subseteq \cdots \subseteq A_k)$.

(iii) One has

$$\overline{\pi}_i(y_{M_i}) = \frac{y_{\{i+1\}}y_{M_i} - y_{\{i+1\}}s_i(y_{M_i})}{y_{\{i\}} - y_{\{i+1\}}}.$$

4.2.2 Basic properties

We define the *invariant algebra* $\mathbb{F}[\mathcal{B}_n]^{H_n(0)}$ of the $H_n(0)$-action on $\mathbb{F}[\mathcal{B}_n]$ to be the trivial isotypic component of $\mathbb{F}[\mathcal{B}_n]$ as an $H_n(0)$-module, namely

$$\begin{aligned} \mathbb{F}[\mathcal{B}_n]^{H_n(0)} &:= \{f \in \mathbb{F}[\mathcal{B}_n] : \pi_i f = f,\ i = 1, \ldots, n-1\} \\ &= \{f \in \mathbb{F}[\mathcal{B}_n] : \overline{\pi}_i f = 0,\ i = 1, \ldots, n-1\}. \end{aligned}$$

This is an analogue of the invariant algebra $\mathbb{F}[\mathcal{B}_n]^{\mathfrak{S}_n}$, which equals $\mathbb{F}[\Theta]$ by Proposition 4.1.1, and we show that they are actually the same.

Proposition 4.2.2. *The invariant algebra* $\mathbb{F}[\mathcal{B}_n]^{H_n(0)}$ *equals* $\mathbb{F}[\Theta]$.

Proof. Let $i \in [n-1]$. We denote by $\mathcal{M}_1$, $\mathcal{M}_2$, and $\mathcal{M}_3$ the sets of all multichains M in $\mathcal{B}_n$ with $p_i(M) < p_{i+1}(M)$, $p_i(M) = p_{i+1}(M)$, and $p_i(M) > p_{i+1}(M)$, respectively. The action of s_i pointwise fixes $\mathcal{M}_2$ and bijectively sends $\mathcal{M}_1$ to $\mathcal{M}_3$. It follows from (4.5) that $\overline{\pi}_i(f) = 0$ if and only if

$$f = \sum_{M \in \mathcal{M}_1} a_M(y_M + y_{s_iM}) + \sum_{M \in \mathcal{M}_2} b_M y_M, \quad a, b \in \mathbb{F}.$$

This is also equivalent to $s_i(f) = f$. Therefore $\mathbb{F}[\mathcal{B}_n]^{H_n(0)} = \mathbb{F}[\mathcal{B}_n]^{\mathfrak{S}_n} = \mathbb{F}[\Theta]$. □

The $\mathfrak{S}_n$-action on $\mathbb{F}[\mathcal{B}_n]$ is Θ-linear, and so is the $H_n(0)$-action.

Proposition 4.2.3. *The* $H_n(0)$*-action on* $\mathbb{F}[\mathcal{B}_n]$ *is* Θ*-linear.*

Proof. Let $i \in [n-1]$ and let $M = (A_1 \subseteq \cdots \subseteq A_k)$ be an arbitrary multichain in $\mathcal{B}_n$. Since $\theta_0 = \emptyset$, one has $\overline{\pi}_i(\theta_0 y_M) = \theta_0 \overline{\pi}_i(y_M)$. It remains to show $\overline{\pi}_i(\theta_r y_M) = \theta_r \overline{\pi}_i(y_M)$ for any $r \in [n]$. One has $|A_j| < r \leq |A_{j+1}|$ for some $j \in \{0, 1, \ldots, k\}$, where $A_0 = \emptyset$ and $A_{k+1} = [n]$ by convention. Then

$$\theta_r y_M = \sum_{A \in \mathcal{A}} y_M y_A.$$

where

$$\mathcal{A} := \{A \subseteq [n] : |A| = r,\ A_j \subsetneq A \subseteq A_{j+1}\}.$$

If $A \in \mathcal{A}$ then $y_M y_A = y_{M \cup A}$ where $M \cup A$ is the multichain obtained by inserting A into M. Let $\mathcal{A}_1$ [$\mathcal{A}_2$, $\mathcal{A}_3$, resp.] be the collections of all sets A in $\mathcal{A}$ satisfying

$$p_i(M \cup A) < \ [=, >, \text{resp.}]\ p_{i+1}(M \cup A)$$

We distinguish three cases below.

If $p_i(M) > p_{i+1}(M)$, then $\overline{\pi}_i(y_M) = -y_M$. Assume $A \in \mathcal{A}$. If $i \notin A$ then one has $p_i(M \cup A) > p_{i+1}(M \cup A)$. If $i \in A \subseteq A_{j+1}$ then $p_i(M) > p_{i+1}(M)$ forces $i+1 \in A_j \subseteq A$ and one still has $p_i(M \cup A) > p_{i+1}(M \cup A)$. Hence $\mathcal{A} = \mathcal{A}_3$ which implies

$$\overline{\pi}_i(\theta_r y_M) = -\theta_r y_M = \theta_r \overline{\pi}_i(y_M).$$

If $p_i(M) = p_{i+1}(M)$, then $\overline{\pi}_i(y_M) = 0$ and we need to show $\overline{\pi}_i(\theta_r y_M) = 0$. First assume $A \in \mathcal{A}_1$. Then A contains i but not $i+1$, A_j contains neither, and A_{j+1} contains both. Hence $s_i(A) \in \mathcal{A}_3$ and

$$\overline{\pi}_i(y_{M\cup A}) = s_i(y_{M\cup A}) = y_{s_i(M\cup A)}.$$

Similarly if $A \in \mathcal{A}_3$ then $s_i(A) \in \mathcal{A}_1$ and thus s_i gives an bijection between $\mathcal{A}_1$ and $\mathcal{A}_3$. For any $A \in \mathcal{A}_2$ one has $\pi_i(y_{M\cup A}) = 0$. Therefore

$$\overline{\pi}_i(\theta_r y_M) = \sum_{A\in\mathcal{A}_1} \overline{\pi}_i(y_{M\cup A} + \overline{\pi}_i(y_{M\cup A})) = 0.$$

Here the last equality follows from the relation $\overline{\pi}_i^2 = -\overline{\pi}_i$.

Finally, we consider the case $p_i(M) < p_{i+1}(M)$. Assume $A \in \mathcal{A}$. If $i \in A_j$ then $p_i(M \cup A) < p_{i+1}(M \cup A)$. If $i \notin A_j$, then $i+1 \notin A_{j+1}$ and so $i+1 \notin A$, which implies $p_i(M \cup A) < p_{i+1}(M \cup A)$. Thus $\mathcal{A} = \mathcal{A}_1$ and

$$\overline{\pi}_i(\theta_r y_M) = s_i(\theta_r y_M) = \theta_r s_i(y_M) = \theta_r \overline{\pi}_i(y_M).$$

This completes the proof. $\square$

Therefore the *coinvariant algebra* $\mathbb{F}[\mathcal{B}_n]/(\Theta)$ is a multigraded $H_n(0)$-module, and we will see in the next subsection that it carries the regular representation of $H_n(0)$. This cannot be obtained simply by applying the transfer map τ, since τ is *not* a map of $H_n(0)$-modules (see §4.3.1).

4.2.3 Noncommutative Hall-Littlewood symmetric functions

In this subsection we interpret the noncommutative analogues of the Hall-Littlewood symmetric functions by the $H_n(0)$-action on the Stanley-Reisner ring of the Boolean algebra. We write a partition of n as an increasing sequence $\mu = (0 < \mu_1 \leq \cdots \leq \mu_k)$ of positive integers whose sum is n, and view it as a composition in this way whenever needed. We want to establish a complete noncommutative analogue of Theorem 3.4.1, which states that the polynomial ring $\mathbb{C}[X]$ has a homogeneous $\mathfrak{S}_n$-stable ideal J_μ such that the graded Frobenius characteristic of the $\mathfrak{S}_n$-module $R_\mu = \mathbb{C}[X]/J_\mu$ is the modified Hall-Littlewood symmetric function indexed by μ.

Recall from §3.4 that the ideal J_μ is generated by

$$\{e_i(S) : |S| \geq i > |S| - (\mu'_1 + \cdots + \mu'_{|S|}),\ S \subseteq \{x_1, \ldots, x_n\}\}$$

where $\mu' = (0 \leq \mu'_1 \leq \cdots \leq \mu'_n)$ is the conjugate of the partition μ with zero parts added whenever necessary, and $e_i(S)$ is the i-th elementary symmetric function in the set S of variables. We can work over an arbitrary field $\mathbb{F}$, and still denote by J_μ the ideal of $\mathbb{F}[X]$ with the same generators.

Example 4.2.4. According to Example 3.4.3 and the proof of Proposition 3.4.2, if $\mu = (1^k, n-k)$ is a hook, then the ideal $J_{1^k, n-k}$ is generated by $e_1, \ldots, e_k$ and all the monomials $x_{i_1} \cdots x_{i_{k+1}}$ satisfying $1 \leq i_1 < \cdots < i_{k+1} \leq n$.

Now we consider an arbitrary composition $\alpha = (\alpha_1, \ldots, \alpha_\ell)$. The major index of α is $\mathrm{maj}(\alpha) := \sum_{i \in D(\alpha)} i$, and viewing a partition μ as a composition one has $\mathrm{maj}(\mu) = n(\mu)$. Recall that α^c is the composition of n with $D(\alpha^c) = [n-1] \setminus D(\alpha)$, $\overleftarrow{\alpha} := (\alpha_\ell, \ldots, \alpha_1)$, and $\alpha' := \overleftarrow{\alpha^c} = (\overleftarrow{\alpha})^c$, whose ribbon diagram is the transpose of the ribbon α.

Bergeron and Zabrocki [8] introduced a noncommutative modified Hall-Littlewood symmetric function

$$\widetilde{\mathbf{H}}_\alpha(\mathbf{x}; t) := \sum_{\beta \preccurlyeq \alpha} t^{\mathrm{maj}(\beta)} \mathbf{s}_\beta \quad \text{inside} \quad \mathbf{NSym}[t] \tag{4.6}$$

and a (q,t)-analogue

$$\widetilde{\mathbf{H}}_\alpha(\mathbf{x}; q, t) := \sum_{\beta \models n} t^{c(\alpha,\beta)} q^{c(\alpha', \overleftarrow{\beta})} \mathbf{s}_\beta \quad \text{inside} \quad \mathbf{NSym}[q,t] \tag{4.7}$$

for every composition α, where $\mathbf{s}_\beta$ is the noncommutative ribbon Schur function indexed by β defined in §2.6 and $c(\alpha, \beta) := \sum_{i \in D(\alpha) \cap D(\beta)} i$. In Corollary 3.4.5 we gave a partial representation theoretic interpretation of $\widetilde{\mathbf{H}}_\alpha(\mathbf{x}; t)$ when $\alpha = (1^k, n-k)$ is a hook, using the $H_n(0)$-action on the polynomial ring $\mathbb{F}[X]$ by the Demazure operators.

Now we switch to the Stanley-Reisner ring $\mathbb{F}[\mathcal{B}_n]$ and provide a complete representation theoretic interpretation for $\widetilde{\mathbf{H}}_\alpha(\mathbf{x}; t)$ and $\widetilde{\mathbf{H}}_\alpha(\mathbf{x}; q, t)$. Recall that $\underline{t}^S$ means the product of t_i for all elements i in a multiset $S \subseteq [n-1]$, with repetitions included.

Theorem 4.2.5. *Let α be a composition of n, and let I_α be the ideal of $\mathbb{F}[\mathcal{B}_n]$ generated by*

$$\Theta_\alpha := \{\theta_i : i \in D(\alpha) \cup \{n\}\} \quad \textit{and} \quad \{y_A \subseteq [n] : |A| \notin D(\alpha) \cup \{n\}\}.$$

Then one has an isomorphism $\mathbb{F}[\mathcal{B}_n]/I_\alpha \cong \mathbb{F}[\mathcal{B}_\alpha]/(\Theta_\alpha)$ *of multigraded* $\mathbb{F}$*-algebras,* $\mathfrak{S}_n$*-modules, and* $H_n(0)$*-modules. In addition, the multigraded noncommutative characteristic of* $\mathbb{F}[\mathcal{B}_n]/I_\alpha$ *equals*

$$\widetilde{\mathbf{H}}_\alpha(\mathbf{x}; t_1, \ldots, t_{n-1}) := \sum_{\beta \preccurlyeq \alpha} \underline{t}^{D(\beta)} \mathbf{s}_\beta \quad \text{inside} \quad \mathbf{NSym}[t_1, \ldots, t_{n-1}].$$

Proof. In §4.1.3 we defined a projection $\phi_\alpha : \mathbb{F}[\mathcal{B}_n] \twoheadrightarrow \mathbb{F}[\mathcal{B}_\alpha]$ which induces an isomorphism

$$\mathbb{F}[\mathcal{B}_n]/(A \subseteq [n] : |A| \notin D(\alpha) \cup \{n\}) \cong \mathbb{F}[\mathcal{B}_\alpha]$$

of multigraded $\mathbb{F}$-algebras and $\mathfrak{S}_n$-modules. By definition, the $H_n(0)$-action also preserves ϕ_α. Since the actions of $\mathfrak{S}_n$ and $H_n(0)$ are both Θ_α-linear, we can take a further quotient by the ideal generated by Θ_α and obtain the desired isomorphism $\mathbb{F}[\mathcal{B}_n]/I_\alpha \cong \mathbb{F}[\mathcal{B}_\alpha]/(\Theta_\alpha)$ of multigraded $\mathbb{F}$-algebras, $\mathfrak{S}_n$-modules, and $H_n(0)$-modules.

Since $\mathbb{F}[\mathcal{B}_\alpha]/(\Theta_\alpha)$ has an $\mathbb{F}$-basis of the descent monomials Y_w for all $w \in \mathfrak{S}^\alpha$, it equals the direct sum of Q_β, the $\mathbb{F}$-span of $\{Y_w : D(w) = D(\beta)\}$, for all $\beta \preccurlyeq \alpha$; each Q_β has homogeneous multigrading $\underline{t}^{D(\beta)}$. The projective indecomposable $H_n(0)$-module $\mathbf{P}_\beta = H_n(0)\overline{\pi}_{w_0(\beta)}\pi_{w_0(\beta^c)}$ has an $\mathbb{F}$-basis

$$\left\{\overline{\pi}_w \pi_{w_0(\beta^c)} : D(w) = D(\beta)\right\}.$$

Thus one has an vector space isomorphism $Q_\beta \cong \mathbf{P}_\beta$ via $Y_w \mapsto \overline{\pi}_w \pi_{w_0(\beta^c)}$. We want to show that this isomorphism is $H_n(0)$-equivariant. Let $i \in [n-1]$ be arbitrary. Suppose that $D(w) = D(\beta)$, and let M be the chain of the sets $\{w(1), \ldots, w(j)\}$ for all $j \in D(w)$. Then $\alpha(M) = \beta$ and $\sigma(M) = w$. We distinguish three cases below and use (4.1).

If $p_i(M) > p_{i+1}(M)$, i.e. $i \in D(w^{-1})$, then one has $\overline{\pi}_i(Y_w) = -Y_w$ and $\overline{\pi}_i \overline{\pi}_w \pi_{w_0(\beta^c)} = -\overline{\pi}_w \pi_{w_0(\beta^c)}$.

If $p_i(M) = p_{i+1}(M)$, i.e. $i \notin D(w^{-1})$ and $D(s_i w) \not\subseteq D(\beta)$, then $\overline{\pi}_i(Y_w) = 0$ and there exists $j \in D(s_i w) \setminus D(\beta)$ such that $\overline{\pi}_i \overline{\pi}_w \pi_{w_0(\beta^c)} = \overline{\pi}_w \overline{\pi}_j \pi_{w_0(\beta^a c)} = 0$ since $\overline{\pi}_j \pi_j = 0$.

If $p_i(M) < p_{i+1}(M)$, i.e. $i \notin D(w^{-1})$ and $D(s_i w) \subseteq D(\beta)$, then $\overline{\pi}_i(Y_w) = y_{s_i w}$ and $\overline{\pi}_i \overline{\pi}_w \pi_{w_0(\beta^c)} = \overline{\pi}_{s_i w} \pi_{w_0(\beta^c)}$.

Therefore $Q_\beta \cong \mathbf{P}_\beta$ is an isomorphism of $H_n(0)$-modules for all $\beta \preccurlyeq \alpha$. It follows that the multigraded noncommutative characteristic of $\mathbb{F}[\mathcal{B}_n]/I_\alpha \cong \mathbb{F}[\mathcal{B}_\alpha]/(\Theta_\alpha)$ is $\widetilde{\mathbf{H}}_\alpha(\mathbf{x}; t_1, \ldots, t_{n-1})$. $\square$

It is easy to see $\widetilde{\mathbf{H}}_\alpha(\mathbf{x};t) = \widetilde{\mathbf{H}}_\alpha(\mathbf{x};t,t^2,\ldots,t^{n-1})$. Thus Theorem 4.2.5 provides a representation theoretic interpretation of $\widetilde{\mathbf{H}}_\alpha(\mathbf{x};t)$ for all compositions α, and can be viewed as a noncommutative analogue of Theorem 3.4.1.

Remark 4.2.6. The proof of Theorem 4.2.5 is actually simpler than the proof of our partial interpretation Corollary 3.4.5 for $\widetilde{\mathbf{H}}_\alpha(\mathbf{x};t)$. This is because $\overline{\pi}_i$ sends a descent monomial in $\mathbb{F}[\mathcal{B}_n]$ to either 0 or ± 1 times a descent monomial, but sends a descent monomial in $\mathbb{F}[X]$ to a polynomial in general (whose leading term is still a descent monomial). We view the Stanley-Reisner ring $\mathbb{F}[\mathcal{B}_n]$ (or $\mathbb{F}[\mathcal{B}_n^*] \cong \mathbb{F}[\mathcal{B}_n]/(\emptyset)$) as a $q=0$ analogue of the polynomial ring $\mathbb{F}[X]$. For an odd (i.e. $q=-1$) analogue, see Lauda and Russell [44].

Remark 4.2.7. When $\alpha = (1^k, n-k)$ is a hook, one can check that the ideal $I_{1^k,n-k}$ of $\mathbb{F}[\mathcal{B}_n]$ has generators $\theta_1,\ldots,\theta_k$ and all y_A with $A \subseteq [n]$ and $|A| \notin [k]$. One can also check that the images of these generators under the transfer map τ are the Tanisaki generators for the ideal $J_{1^k,n-k}$ of $\mathbb{F}[X]$, although $\tau(I_{1^k,n-k}) \neq J_{1^k,n-k}$.

Remark 4.2.8. One sees that the coinvariant algebra $\mathbb{F}[\mathcal{B}_n]/(\Theta)$ carries the regular representation of $H_n(0)$, as its multigraded noncommutative characteristic equals

$$\widetilde{\mathbf{H}}_{1^n}(\mathbf{x};t_1,\ldots,t_{n-1}) := \sum_{\beta \models n} \underline{t}^{D(\beta)} \mathbf{s}_\beta.$$

If we take $t_i = t^i$ for all $i \in D(\alpha)$, and $t_i = q^{n-i}$ for all $i \in D(\alpha^c)$ in $\widetilde{\mathbf{H}}_{1^n}(\mathbf{x};t_1,\ldots,t_{n-1})$, then we obtain the (q,t)-analogue $\widetilde{\mathbf{H}}_\alpha(\mathbf{x};q,t)$ from (4.7).

Hivert, Lascoux, and Thibon [30] defined a family of noncommutative symmetric functions on multiple parameters q_i and t_i, which are similar to but different from the family $\left\{\widetilde{\mathbf{H}}_\alpha(\mathbf{x};q,t)\right\}$. A common generalization of these two families of noncommutative symmetric functions was discovered recently by Lascoux, Novelli, and Thibon [42], namely a family $\{P_\alpha\}$ of noncommutative symmetric functions having parameters associated with paths in binary trees.

In fact, one recovers P_α from $\widetilde{\mathbf{H}}_{1^n}(\mathbf{x};t_1,\ldots,t_{n-1})$, the noncommutative characteristic of the coinvariant algebra $\mathbb{F}[\mathcal{B}_n]/(\Theta)$. For any composition α of n, let $u(\alpha) = u_1\cdots u_{n-1}$ be the Boolean word such that $u_i = 1$ if $i \in D(\alpha)$ and $u_i = 0$ otherwise. Let $y_{u_{1\ldots i}}$ be a parameter indexed by the Boolean word $u_1\cdots u_i$. It follows from the definition of

P_α [42, (31)] that

$$P_\alpha = \sum_{\beta \models n} \left(\prod_{i \in D(\beta)} y_{u_{1\ldots i}} \right) \mathbf{s}_\beta.$$

Then taking $t_i = y_{u_{1\ldots i}}$ one has $\widetilde{\mathbf{H}}_{1^n}(\mathbf{x}; y_{u_{1\ldots 1}}, \ldots, y_{u_{1\ldots n-1}}) = P_\alpha$. For example, when $\alpha = 211$ one has $u(\alpha) = 011$ and

$$\begin{aligned} \widetilde{\mathbf{H}}_{1111}(\mathbf{x}; y_0, y_{01}, y_{011}) = \mathbf{s}_4 + y_{011}\mathbf{s}_{31} + y_{01}\mathbf{s}_{22} + y_{01}y_{011}\mathbf{s}_{211} + y_0\mathbf{s}_{13} \\ + y_0y_{011}\mathbf{s}_{121} + y_0y_{01}\mathbf{s}_{112} + y_0y_{01}y_{011}\mathbf{s}_{1111} = P_{211}. \end{aligned}$$

4.2.4 Properties of $\mathbf{H}_\alpha(\mathbf{x}; t_1, \ldots, t_{n-1})$

The multigraded noncommutative characteristic $\widetilde{\mathbf{H}}_\alpha(\mathbf{x}; t_1, \ldots, t_{n-1})$, where $\alpha \models n$, is the modified version of

$$\mathbf{H}_\alpha = \mathbf{H}_\alpha(\mathbf{x}; t_1, \ldots, t_{n-1}) := \sum_{\beta \preccurlyeq \alpha} \underline{t}^{D(\alpha) \setminus D(\beta)} \mathbf{s}_\beta$$

which is a multivariate noncommutative analogue of the Hall-Littlewood symmetric functions inside $\mathbf{NSym}[t_1, \ldots, t_{n-1}]$. We show below that these functions satisfy similar properties to those given in [8] for $\mathbf{H}_\alpha(\mathbf{x}; t)$; taking $t_i = t^i$ for all $i \in [n-1]$ one recovers the corresponding results in [8].

It is easy to see $\mathbf{H}_\alpha(0, \ldots, 0) = \mathbf{s}_\alpha$ and $\mathbf{H}_\alpha(1, \ldots, 1) = \mathbf{h}_\alpha$. Let $\mathbf{NSym}_n$ be the n-th homogeneous component of $\mathbf{NSym}$, which has bases $\{\mathbf{s}_\alpha : \alpha \models n\}$ and $\{\mathbf{h}_\alpha : \alpha \models n\}$. Then $\{\mathbf{H}_\alpha : \alpha \models n\}$ gives a basis for $\mathbf{NSym}_n[t_1, \ldots, t_{n-1}]$, since $\mathbf{H}_\alpha$ has leading term $\mathbf{s}_\alpha$ under the partial order $\preccurlyeq$ for compositions of n. It follows that $\bigsqcup_{n \geq 0} \{\mathbf{H}_\alpha : \alpha \models n\}$ is a basis for $\mathbf{NSym}[t_1, t_2, \ldots]$.

Bergeron and Zabrocki [8] defined an inner product on $\mathbf{NSym}$ such that the basis $\{\mathbf{s}_\alpha\}$ is "semi-self dual", namely $\langle \mathbf{s}_\alpha, \mathbf{s}_\beta \rangle := (-1)^{|\alpha| + \ell(\alpha)} \delta_{\alpha, \beta^c}$ where δ is the Kronecker delta. They showed that the same result holds for $\{\mathbf{h}_\alpha\}$ and $\{\mathbf{H}_\alpha(\mathbf{x}; t)\}$. Here we prove a multivariate version.

Proposition 4.2.9. *One has $\langle \mathbf{H}_\alpha, \mathbf{H}_\beta \rangle = (-1)^{|\alpha| + \ell(\alpha)} \delta_{\alpha, \beta^c}$ for any pair of compositions α and β.*

Proof. By definition, one has

$$\langle \mathbf{H}_\alpha, \mathbf{H}_\beta \rangle = \sum_{\alpha' \preccurlyeq \alpha} \underline{t}^{D(\alpha)\setminus D(\alpha')} \sum_{\beta' \preccurlyeq \beta} \underline{t}^{D(\beta)\setminus D(\beta')} \langle \mathbf{s}_{\alpha'}, \mathbf{s}_{\beta'} \rangle.$$

If $|\alpha| \neq |\beta|$ then $\langle \mathbf{s}_{\alpha'}, \mathbf{s}_{\beta'} \rangle = 0$ for all $\alpha' \preccurlyeq \alpha$ and $\beta' \preccurlyeq \beta$. Assume $|\alpha| = |\beta| = n$ below.

If $D(\alpha) \cup D(\beta) \neq [n-1]$ then again one has $\langle \mathbf{s}_{\alpha'}, \mathbf{s}_{\beta'} \rangle = 0$ for all $\alpha' \preccurlyeq \alpha$ and $\beta' \preccurlyeq \beta$.

If $\alpha = \beta^c$ then the right hand side contains only one nonzero term $\langle \mathbf{s}_\alpha, \mathbf{s}_\beta \rangle$.

If $D(\alpha) \cap D(\beta) \neq \emptyset$ then taking $E = D(\alpha) \setminus D(\alpha')$ we write the right hand side as

$$\sum_{E \subseteq D(\alpha) \cap D(\beta)} (-1)^{n+\ell(\alpha)-|E|} \, \underline{t}^{D(\alpha)\cap D(\beta)} = 0.$$

This completes the proof. $\square$

We also give a product formula for $\{\mathbf{H}_\alpha\}$, generalizing the product formula for $\{\mathbf{H}_\alpha(\mathbf{x};t)\}$ given by Bergeron and Zabrocki [8]. Recall from 2.7 that

$$\mathbf{s}_\alpha \mathbf{s}_\beta = \mathbf{s}_{\alpha\beta} + \mathbf{s}_{\alpha \rhd \beta}$$

for all compositions α and β, where

$$\alpha\beta := (\alpha_1, \ldots, \alpha_\ell, \beta_1, \ldots, \beta_k),$$

$$\alpha \rhd \beta := (\alpha_1, \ldots, \alpha_{\ell-1}, \alpha_\ell + \beta_1, \beta_2, \ldots, \beta_k).$$

Proposition 4.2.10. *For any compositions α and β, one has the product formula*

$$\mathbf{H}_\alpha \cdot \mathbf{H}_\beta = \sum_{\gamma \preccurlyeq \beta} \left(\prod_{i \in D(\beta)\setminus D(\gamma)} (t_i - t_{|\alpha|+i}) \right) \left(\mathbf{H}_{\alpha\gamma} + (1 - t_{|\alpha|}) \mathbf{H}_{\alpha \rhd \gamma} \right).$$

Proof. Let $\gamma \preccurlyeq \beta$. If $\delta \preccurlyeq \alpha\gamma$ then there exists a unique pair of compositions $\alpha' \preccurlyeq \alpha$ and $\gamma' \preccurlyeq \gamma$ such that $\delta = \alpha'\gamma'$ or $\delta = \alpha' \rhd \gamma'$. If $\delta \preccurlyeq \alpha \rhd \gamma$ then there exists a unique pair of compositions $\alpha' \preccurlyeq \alpha$ and $\gamma' \preccurlyeq \gamma$ such that $\delta = \alpha' \rhd \gamma'$. Thus

$$\mathbf{H}_{\alpha\gamma} + (1 - t_{|\alpha|}) \mathbf{H}_{\alpha \rhd \gamma} = \sum_{\substack{\alpha' \preccurlyeq \alpha \\ \gamma' \preccurlyeq \gamma}} \Big(\underline{t}^{D(\alpha\gamma)\setminus D(\alpha'\gamma')} \mathbf{s}_{\alpha'\gamma'} + \underline{t}^{D(\alpha\gamma)\setminus D(\alpha' \rhd \gamma')} \mathbf{s}_{\alpha' \rhd \gamma'}$$
$$+ (1 - t_{|\alpha|}) \, \underline{t}^{D(\alpha \rhd \gamma)\setminus D(\alpha' \rhd \gamma')} \mathbf{s}_{\alpha' \rhd \gamma'} \Big).$$

Since $D(\alpha\gamma) = D(\alpha \triangleright \gamma) \sqcup \{|\alpha|\}$ and $D(\alpha\gamma) \setminus D(\alpha'\gamma') = D(\alpha \triangleright \gamma) \setminus D(\alpha' \triangleright \gamma')$, it follows that

$$\mathbf{H}_{\alpha\gamma} + (1 - t_{|\alpha|})\mathbf{H}_{\alpha\triangleright\gamma} = \sum_{\substack{\alpha' \preccurlyeq \alpha \\ \gamma' \preccurlyeq \gamma}} \underline{t}^{D(\alpha\gamma)\setminus D(\alpha'\gamma')}(\mathbf{s}_{\alpha'\gamma'} + \mathbf{s}_{\alpha'\triangleright\gamma'}).$$

Note that $\mathbf{s}_{\alpha'\gamma'} + \mathbf{s}_{\alpha'\triangleright\gamma'} = \mathbf{s}_{\alpha'}\mathbf{s}_{\gamma'}$, and

$$D(\alpha\gamma) \setminus D(\alpha'\gamma') = (D(\alpha) \setminus D(\alpha')) \sqcup \{\, |\alpha| + i : i \in D(\gamma) \setminus D(\gamma')\}.$$

Thus the right hand side of the product formula equals

$$\sum_{\substack{\alpha' \preccurlyeq \alpha \\ \gamma' \preccurlyeq \beta}} \underline{t}^{D(\alpha)\setminus D(\alpha')}\mathbf{s}_{\alpha'}\mathbf{s}_{\gamma'} \sum_{\gamma' \preccurlyeq \gamma \preccurlyeq \beta} \left(\prod_{i \in D(\beta)\setminus D(\gamma)} (t_i - t_{|\alpha|+i}) \right) \underline{t}^{|\alpha|+D(\gamma)\setminus D(\gamma')}$$

where $\underline{t}^{|\alpha|+S} := \prod_{i\in S} t_{|\alpha|+i}$. Since the interval $[\gamma', \beta]$ is isomorphic to the Boolean algebra of the subsets of $D(\beta) \setminus D(\gamma')$, one sees that

$$\sum_{\gamma' \preccurlyeq \gamma \preccurlyeq \beta} \left(\prod_{i \in D(\beta)\setminus D(\gamma)} (t_i - t_{|\alpha|+i}) \right) \underline{t}^{|\alpha|+D(\gamma)\setminus D(\gamma')} = \underline{t}^{D(\beta)\setminus D(\gamma')}.$$

Therefore the right-hand side of the product formula is equal to

$$\sum_{\alpha' \preccurlyeq \alpha} \underline{t}^{D(\alpha)\setminus D(\alpha')}\mathbf{s}_{\alpha'} \sum_{\gamma' \preccurlyeq \beta} \underline{t}^{D(\beta)\setminus D(\gamma')}\mathbf{s}_{\gamma'} = \mathbf{H}_\alpha \cdot \mathbf{H}_\beta.$$

The proof is complete. □

Remark 4.2.11. One recovers the product formula $\mathbf{s}_\alpha\mathbf{s}_\beta = \mathbf{s}_{\alpha\beta} + \mathbf{s}_{\alpha\triangleright\beta}$ from the above proposition by using $\mathbf{s}_\alpha = \mathbf{H}_\alpha(\mathbf{x}; 0, \ldots, 0)$ and $\mathbf{s}_\beta = \mathbf{H}_\beta(\mathbf{x}; 0, \ldots, 0)$.

Corollary 4.2.12. *Let α and β be two compositions. Then*

$$\mathbf{H}_\alpha(\mathbf{x}; t_1, \ldots, t_{n-1})\mathbf{H}_\beta(\mathbf{x}; t|n) = \mathbf{H}_{\alpha\beta}(t|n)$$

where $n = |\alpha|$ and $t|n := \{t_1, \ldots, t_{n-1}, 1, t_1, \ldots, t_{n-1}, 1, \ldots\}$. In particular, if ζ is an n-th root of unity then $\mathbf{H}_\alpha(\mathbf{x}; \zeta)\mathbf{H}_\beta(\mathbf{x}; \zeta) = \mathbf{H}_{\alpha\beta}(\mathbf{x}; \zeta)$ (see Bergeron and Zabrocki [8]).

Proof. The result follows immediately from the above proposition. □

4.2.5 Quasisymmetric characteristic

In this subsection we use the two encodings given in §4.1.2 for the multichains in $\mathcal{B}_n$ to study the quasisymmetric characteristic of the Stanley-Reisner ring $\mathbb{F}[\mathcal{B}_n]$.

Lemma 4.2.13. *Let α be a weak composition of n. Then the α-homogeneous component $\mathbb{F}[\mathcal{B}_n]_\alpha$ of the Stanley-Reisner ring $\mathbb{F}[\mathcal{B}_n]$ is an $H_n(0)$-submodule of $\mathbb{F}[\mathcal{B}_n]$ with homogeneous multigrading $\underline{t}^{D(\alpha)}$ and isomorphic to the cyclic module $H_n(0)\pi_{w_0(\alpha^c)}$, where α^c is the composition of n with descent set $[n-1]\setminus D(\alpha)$.*

Proof. It is not hard to check that $H_n(0)\pi_{w_0(\alpha^c)}$ has an $\mathbb{F}$-basis $\{\overline{\pi}_w\pi_{w_0(\alpha^c)} : w\in\mathfrak{S}^\alpha\}$. For any $w\in\mathfrak{S}^\alpha$, the $H_n(0)$-action is given by

$$\overline{\pi}_i\overline{\pi}_w\pi_{w_0(\alpha^c)} = \begin{cases} -\overline{\pi}_w\pi_{w_0(\alpha^c)} & \text{if } i\in D(w^{-1}),\\ 0, & \text{if } i\notin D(w^{-1}),\ s_iw\notin\mathfrak{S}^\alpha,\\ \overline{\pi}_{s_iw}\pi_{w_0(\alpha^c)}, & \text{if } i\notin D(w^{-1}),\ s_iw\in\mathfrak{S}^\alpha. \end{cases}$$

On the other hand, if M is a multichain of $\mathcal{B}_n$ with $\alpha(M)=\alpha$, then one has $r(M)=D(\alpha)$ and $\sigma(M)\in\mathfrak{S}^\alpha$. It follows from (4.1) that $\mathbb{F}[\mathcal{B}_n]_\alpha\cong H_n(0)\pi_{w_0(\alpha^c)}$ via $y_M\mapsto\overline{\pi}_{\sigma(M)}\pi_{w_0(\alpha^c)}$. □

Since every homogeneous component $\mathbb{F}[\mathcal{B}_n]_\alpha$ is a cyclic multigraded $H_n(0)$-module, we get an $\mathbb{N}\times\mathbb{N}^{n+1}$-multigraded quasisymmetric characteristic

$$\mathrm{Ch}_{q,\underline{t}}(\mathbb{F}[\mathcal{B}_n]_\alpha) = \sum_{w\in\mathfrak{S}^\alpha} q^{\mathrm{inv}(w)}\underline{t}^{D(\alpha)}F_{D(w^{-1})} \tag{4.8}$$

where q keeps track of the length filtration and $\underline{t}$ keeps track of the multigrading of $\mathbb{F}[\mathcal{B}_n]_\alpha$. This defines an $\mathbb{N}\times\mathbb{N}^{n+1}$-multigraded quasisymmetric characteristic for the Stanley-Reisner ring $\mathbb{F}[\mathcal{B}_n]$.

Theorem 4.2.14. *The $\mathbb{N}\times\mathbb{N}^{n+1}$-multigraded quasisymmetric characteristic of $\mathbb{F}[\mathcal{B}_n]$ is*

$$\begin{aligned}\mathrm{Ch}_{q,\underline{t}}(\mathbb{F}[\mathcal{B}_n]) &= \sum_{k\geq 0}\sum_{\alpha\in\mathrm{Com}(n,k+1)}\underline{t}^{D(\alpha)}\sum_{w\in\mathfrak{S}^\alpha} q^{\mathrm{inv}(w)}F_{D(w^{-1})}\\ &= \sum_{w\in\mathfrak{S}_n}\frac{q^{\mathrm{inv}(w)}\underline{t}^{D(w)}F_{D(w^{-1})}}{\prod_{0\leq i\leq n}(1-t_i)}\\ &= \sum_{k\geq 0}\sum_{\mathbf{p}\in[k+1]^n} t_{p'_1}\cdots t_{p'_k}q^{\mathrm{inv}(\mathbf{p})}F_{D(\mathbf{p})}.\end{aligned}$$

Proof. The first expression of $\mathrm{Ch}_{q,\underline{t}}(\mathbb{F}[\mathcal{B}_n])$ follows immediately from (4.8).

To see the second expression, recall that $\mathbb{F}[\mathcal{B}_n]$ is a free $\mathbb{F}[\Theta]$-module on the descent basis $\{Y_w : w \in \mathfrak{S}_n\}$, and the $H_n(0)$-action on $\mathbb{F}[\mathcal{B}_n]$ is $\mathbb{F}[\Theta]$-linear. If $a_0, \ldots, a_n$ are nonnegative integers and M is a multichain in $\mathcal{B}_n$, then one sees that $\theta_0^{a_0} \cdots \theta_n^{a_n} y_M$ is the sum of $y_{M'}$ for all the multichains M' refining M and having rank multiset $r(M') = r(M) \cup \{0^{a_0}, \ldots, n^{a_n}\}$. Thus for any $w \in \mathfrak{S}_n$, the element $\theta_0^{a_0} \cdots \theta_n^{a_n} Y_w$ has leading term

$$\prod_{i \in D(w) \cup \{0^{a_0}, \ldots, n^{a_n}\}} y_{\{w(1), \ldots, w(i)\}}.$$

It follows that $\theta_0^{a_0} \cdots \theta_n^{a_n} Y_w$ has length-grading $q^{\mathrm{inv}(w)}$. Then one has

$$\begin{aligned} \mathrm{Ch}_{q,\underline{t}}\left(\mathbb{F}[\mathcal{B}_n]\right) &= \mathrm{Hilb}\left(\mathbb{F}[\Theta]; \underline{t}\right) \sum_{w \in \mathfrak{S}_n} q^{\mathrm{inv}(w)} \underline{t}^{D(w)} F_{D(w^{-1})} \\ &= \sum_{w \in \mathfrak{S}_n} \frac{q^{\mathrm{inv}(w)} \underline{t}^{D(w)} F_{D(w^{-1})}}{\prod_{0 \le i \le n} (1 - t_i)}. \end{aligned}$$

Finally we encode a multichain M of length k in $\mathcal{B}_n$ by $p(M) = \mathbf{p} \in [k+1]^n$. The $H_n(0)$-action in terms of this encoding is equivalent to the first one via (4.1). One has $D(\alpha(M))$ equals the multiset of $p'_1, \ldots, p'_k$ and $\mathrm{inv}(\sigma(M)) = \mathrm{inv}(p(M))$. Hence we get the third expression of $\mathrm{Ch}_{q,\underline{t}}(\mathbb{F}[\mathcal{B}_n])$. $\square$

4.2.6 Applications to permutation statistics

We explain here how Theorem 4.2.14 specializes to a result of Garsia and Gessel [25, Theorem 2.2] on the multivariate generating function of the permutation statistics $\mathrm{inv}(w)$, $\mathrm{maj}(w)$, $\mathrm{des}(w)$, $\mathrm{maj}(w^{-1})$, and $\mathrm{des}(w^{-1})$ for all $w \in \mathfrak{S}_n$. First recall that

$$F_\alpha = \sum_{\substack{i_1 \ge \cdots \ge i_n \ge 1 \\ i \in D(\alpha) \Rightarrow i_j > i_{j+1}}} x_{i_1} \cdots x_{i_n}, \quad \forall \alpha \models n.$$

Given a nonnegative integer ℓ, let $\mathbf{ps}_{q;\ell}$ be the linear transformation from formal power series in $x_1, x_2, \ldots$ to formal power series in q, defined by $\mathbf{ps}_{q;\ell}(x_i) = q^{i-1}$ for $i = 1, \ldots, \ell$, and $\mathbf{ps}_{q;\ell}(x_i) = 0$ for all $i > \ell$; similarly, $\mathbf{ps}_{q;\infty}$ is defined by $\mathbf{ps}_{q;\infty}(x_i) = q^{i-1}$ for all $i = 1, 2, \ldots$. It is well known (see Stanley [56, Lemma 7.19.10]) that

$$\mathbf{ps}_{q;\infty}(F_\alpha) = \frac{q^{\mathrm{maj}(\alpha)}}{(1-q)\cdots(1-q^n)}.$$

Let $(u;q)_n := (1-u)(1-qu)(1-q^2u)\cdots(1-q^nu)$. It is also not hard to check (see Gessel and Reutenauer [28, Lemma 5.2]) that

$$\sum_{\ell\geq 0} u^\ell \mathbf{ps}_{q;\ell+1}(F_\alpha) = \frac{q^{\mathrm{maj}(\alpha)}u^{\mathrm{des}(\alpha)}}{(u;q)_n}.$$

A *bipartite partition* is a pair of weak compositions $\lambda = (\lambda_1,\dots,\lambda_n)$ and $\mu = (\mu_1,\dots,\mu_n)$ satisfying the conditions $\lambda_1 \geq \cdots \geq \lambda_n$ and $\lambda_i = \lambda_{i+1} \Rightarrow \mu_i \geq \mu_{i+1}$ (so the pairs of nonnegative integers $(\lambda_1,\mu_1),\dots,(\lambda_n,\mu_n)$ are lexicographically ordered). Let $B(\ell,k)$ be the set of bipartite partitions (λ,μ) such that $\max(\lambda)\leq \ell$ and $\max(\mu)\leq k$, where $\max(\mu) := \max\{\mu_1,\dots,\mu_n\}$ and similarly for $\max(\lambda)$.

Corollary 4.2.15 (Garsia and Gessel [25])**.**

$$\frac{\sum_{w\in\mathfrak{S}_n} q_0^{\mathrm{inv}(w)} q_1^{\mathrm{maj}(w^{-1})} u_1^{\mathrm{des}(w^{-1})} q_2^{\mathrm{maj}(w)} u_2^{\mathrm{des}(w)}}{(u_1;q_1)_n(u_2;q_2)_n} = \sum_{\ell,k\geq 0} u_1^\ell u_2^k \sum_{(\lambda,\mu)\in B(\ell,k)} q_0^{\mathrm{inv}(\mu)} q_1^{|\lambda|} q_2^{|\mu|}.$$

Proof. Theorem 4.2.14 gives the equality

$$\sum_{w\in\mathfrak{S}_n} \frac{q_0^{\mathrm{inv}(w)} \underline{t}^{D(w)} F_{D(w^{-1})}}{\prod_{0\leq i\leq n}(1-t_i)} = \sum_{k\geq 0}\sum_{\mathbf{p}\in[k+1]^n} t_{p_1'}\cdots t_{p_k'} q_0^{\mathrm{inv}(\mathbf{p})} F_{D(\mathbf{p})}. \tag{4.9}$$

Applying the linear transformation $\sum_{\ell\geq 0} u_1^\ell \mathbf{ps}_{q_1;\ell+1}$ and also the specialization $t_i = q_2^i u_2$ for $i = 0,1,\dots,n$ to this equality, we obtain

$$\begin{aligned}
&\frac{\sum_{w\in\mathfrak{S}_n} q_0^{\mathrm{inv}(w)} q_1^{\mathrm{maj}(w^{-1})} u_1^{\mathrm{des}(w^{-1})} q_2^{\mathrm{maj}(w)} u_2^{\mathrm{des}(w)}}{(u_1;q_1)_n(u_2;q_2)_n} \\
&= \sum_{k\geq 0} u_2^k \sum_{\mathbf{p}\in[k+1]^n} q_2^{p_1'+\cdots p_k'} q_0^{\mathrm{inv}(\mathbf{p})} \sum_{\ell\geq 0} u_1^\ell \sum_{\substack{\ell\geq\lambda_1\geq\cdots\geq\lambda_n\geq 0\\ j\in D(\mathbf{p})\Rightarrow \lambda_j>\lambda_{j+1}}} q_1^{\lambda_1+\cdots+\lambda_n}.
\end{aligned}$$

Note that $\mathbf{p}\in[k+1]^n$ if and only if $\mu := (k+1-p_1,\dots,k+1-p_n)$ is a weak composition with $\max(\mu)\leq k$, and one has $|\mu| = p_1'+\cdots+p_k'$ by the definition of p_i'. The condition $j\in D(\mathbf{p}) \Rightarrow \lambda_j > \lambda_{j+1}$ is equivalent to $\lambda_i = \lambda_{i+1} \Rightarrow \mu_i \geq \mu_{i+1}$. Thus we can rewrite the right hand side as a sum over $(\lambda,\mu)\in B(\ell,k)$ for all $\ell,k\geq 0$, and then the result follows easily. $\square$

Taking $q_0 = 1$ in Theorem 4.2.14 one has the usual $\mathbb{N}^{n+1}$-multigraded quasisymmetric characteristic $\mathrm{Ch}_{\underline{t}}(\mathbb{F}[\mathcal{B}_n])$. Then applying the same specialization as in the proof of the above corollary, and using the observation

$$\sum_{(\lambda,\mu)\in B(\ell,k)} q_1^{|\lambda|} q_2^{|\mu|} = \prod_{0\le i\le \ell}\prod_{0\le j\le k} \frac{1}{1-zq_1^i q_2^j}\Bigg|_{z^n}$$

where $f|_{z^n}$ is the coefficient of z^n in f, one can get another result of Garsia and Gessel [25]:

$$\frac{\sum_{w\in\mathfrak{S}_n} q_1^{\mathrm{maj}(w^{-1})} u_1^{\mathrm{des}(w^{-1})} q_2^{\mathrm{maj}(w)} u_2^{\mathrm{des}(w)}}{(u_1;q_1)_n (u_2;q_2)_n} = \sum_{\ell,k\ge 0} u_1^\ell u_2^k \prod_{0\le i\le \ell}\prod_{0\le j\le k} \frac{1}{1-zq_1^i q_2^j}\Bigg|_{z^n}.$$

A further specialization of Theorem 4.2.14 gives a well known result which is often attributed to Carlitz [13] but actually dates back to MacMahon [45, Volume 2, Chapter 4].

Corollary 4.2.16 (MacMahon-Carlitz)**.** *Let* $[k+1]_q := 1+q+q^2+\cdots+q^k$. *Then*

$$\frac{\sum_{w\in\mathfrak{S}_n} q^{\mathrm{maj}(w)} u^{\mathrm{des}(w)}}{(u;q)_n} = \sum_{k\ge 0} ([k+1]_q)^n u^k.$$

Proof. Taking $q_0 = 1$, $t_i = q^i u$ for all $i = 0, 1, \ldots, n$, and $F_I = 1$ for all $I \subseteq [n-1]$ in (4.9, we get

$$\frac{\sum_{w\in\mathfrak{S}_n} q^{\mathrm{maj}(w)} u^{\mathrm{des}(w)}}{(u;q)_n} = \sum_{k\ge 0} u^k \sum_{\mathbf{p}\in[k+1]^n} q^{p_1'+\cdots+p_k'}.$$

Then using Equation (4.2) we establish this corollary. □

Theorem 4.2.14 also implies the following result, which was obtained by Adin, Brenti, and Roichman [1] from the Hilbert series of the coinvariant algebra $\mathbb{F}[X]/(\mathbb{F}[X]_+^{\mathfrak{S}_n})$.

Corollary 4.2.17 (Adin, Brenti, and Roichman [1])**.** *Let* $\mathrm{Par}(n)$ *be the set of all weak partitions* $\lambda = (\lambda_1, \ldots, \lambda_n)$ *with* $\lambda_1 \ge \cdots \ge \lambda_n \ge 0$, *and let* $m(\lambda) = (m_0(\lambda), m_1(\lambda), \ldots)$, *where*

$$m_j(\lambda) := \#\{1 \le i \le n : \lambda_i = j\}.$$

Then

$$\sum_{\lambda\in\mathrm{Par}(n)} \binom{n}{m(\lambda)} \prod_{i=1}^n q_i^{\lambda_i} = \frac{\sum_{w\in\mathfrak{S}_n} \prod_{i\in D(w)} q_1\cdots q_i}{(1-q_1)(1-q_1q_2)\cdots(1-q_1\cdots q_n)}.$$

Proof. Recall that the rank-selected Boolean algebra $\mathcal{B}_n^*$ consists of *nonempty* subsets of $[n]$, and one has an isomorphism $\mathbb{F}[\mathcal{B}_n^*] \cong \mathbb{F}[\mathcal{B}_n]/(\emptyset)$ of $\mathbb{F}$-algebras.

Given an integer $k \geq 0$, the multichains $M = (A_1 \subseteq \cdots \subseteq A_k)$ with $A_1 \neq \emptyset$ are in bijection with the pairs $(\alpha(M), \sigma(M))$ of $\alpha(M) \in \mathrm{Com}_1(n, k+1)$ and $\sigma(M) \in \mathfrak{S}^\alpha$, where

$$\mathrm{Com}_1(n, k+1) := \{(\alpha_1, \ldots, \alpha_{k+1}) \in \mathrm{Com}(n, k+1) : \alpha_1 \geq 1\}.$$

Hence the proof of Theorem 4.2.14 implies that

$$\begin{aligned} \mathrm{Ch}_{\underline{t}}(\mathbb{F}[\mathcal{B}_n^*]) &= \sum_{k\geq 0} \sum_{\alpha \in \mathrm{Com}_1(n,k+1)} \underline{t}^{D(\alpha)} \sum_{w\in\mathfrak{S}^\alpha} F_{D(w^{-1})} \\ &= \frac{\sum_{w\in\mathfrak{S}_n} \underline{t}^{D(w)} F_{D(w^{-1})}}{\prod_{1\leq i\leq n}(1-t_i)}. \end{aligned}$$

Taking $t_i = q_1 \cdots q_i$ for $i = 1, \ldots, n$, and $F_{D(w^{-1})} = 1$ for all $w \in \mathfrak{S}_n$, we obtain

$$\sum_{k\geq 0} \sum_{\alpha\in\mathrm{Com}_1(n,k+1)} \binom{n}{\alpha} \prod_{i\in D(\alpha)} q_1 \cdots q_i = \frac{\sum_{w\in\mathfrak{S}_n} \prod_{i\in D(w)} q_1 \cdots q_i}{(1-q_1)\cdots(1-q_1\cdots q_n)}.$$

Thus it remains to show

$$\sum_{\lambda\in\mathrm{Par}(k,n)} \binom{n}{m(\lambda)} \prod_{i=1}^{n} q_i^{\lambda_i} = \sum_{\alpha\in\mathrm{Com}_1(n,k+1)} \binom{n}{\alpha} \prod_{j\in D(\alpha)} q_1 \cdots q_j$$

where $\mathrm{Par}(k, n) := \{(\lambda_1, \ldots, \lambda_n) \in \mathrm{Par}(n) : \lambda_1 = k\}$, for all $k \geq 0$. This can be established by using the bijection $\lambda \mapsto \alpha(\lambda) := (m_k(\lambda), \ldots, m_0(\lambda))$ between $\mathrm{Par}(k, n)$ and $\mathrm{Com}_1(n, k+1)$. One sees that the multiset $D(\alpha(\lambda))$ is precisely the multiset of column lengths of the Young diagram of λ, and thus

$$\lambda_i = \#\{j \in D(\alpha(\lambda)) : j \geq i\}, \quad \forall i \in [n].$$

This completes the proof. □

4.3 Remarks and questions for future research

4.3.1 Connection with the polynomial ring

We give in §4.1.3 an analogy via the transfer map τ between the rank-selected Stanley-Reisner ring $\mathbb{F}[\mathcal{B}_n^*]$ as a multigraded algebra and $\mathfrak{S}_n$-module and the polynomial ring

$\mathbb{F}[X]$ as a graded algebra and $\mathfrak{S}_n$-module. With our $H_n(0)$-action on $\mathbb{F}[\mathcal{B}_n^*]$ and the usual $H_n(0)$-action on $\mathbb{F}[X]$, the transfer map τ is not an isomorphism of $H_n(0)$-modules: e.g. for $n=2$ one has $\tau(y_1^2)=x_1^2$ but

$$\overline{\pi}_1(y_1^2)=y_2^2, \quad \overline{\pi}_1(x_1^2)=x_2^2+x_1x_2\neq x_2^2=\tau(y_2^2).$$

However, there is still a similar analogy between the multigraded $H_n(0)$-module $\mathbb{F}[\mathcal{B}_n^*]$ and the graded $H_n(0)$-module $\mathbb{F}[X]$. In fact, Theorem 4.2.5 and Theorem 4.2.14 imply

$$\mathbf{ch}_{\underline{t}}(\mathbb{F}[\mathcal{B}_n^*]) = \frac{\sum_{\alpha\models n}\underline{t}^{D(\alpha)}\mathbf{s}_\alpha}{\prod_{1\le i\le n}(1-t_i)},$$

$$\mathrm{Ch}_{q,\underline{t}}(\mathbb{F}[\mathcal{B}_n^*]) = \frac{\sum_{w\in\mathfrak{S}_n} q^{\mathrm{inv}(w)}\underline{t}^{D(w)}F_{D(w^{-1})}}{\prod_{1\le i\le n}(1-t_i)}.$$

They specialize to the graded noncommutative characteristic and bigraded quasisymmetric characteristic of $\mathbb{F}[X]$ via $t_i=t^i$ for $i=1,\dots,n$, as it follows from results in the previous chapter that

$$\mathbf{ch}_t(\mathbb{F}[X]) = \frac{\sum_{\alpha\models n} t^{\mathrm{maj}(\alpha)}\mathbf{s}_\alpha}{\prod_{1\le i\le n}(1-t^i)},$$

$$\mathrm{Ch}_{q,t}(\mathbb{F}[X]) = \frac{\sum_{w\in\mathfrak{S}_n} q^{\mathrm{inv}(w)}t^{\mathrm{maj}(w)}F_{D(w^{-1})}}{\prod_{1\le i\le n}(1-t^i)}.$$

This suggests an isomorphism $\mathbb{F}[\mathcal{B}_n^*]\cong\mathbb{F}[X]$ of graded $H_n(0)$-modules. To explicitly give such an isomorphism, we consider every α-homogeneous component of $\mathbb{F}[\mathcal{B}_n^*]$, which has a basis

$$\left\{ \overline{\pi}_w\left(\prod_{i\in D(\alpha)} y_{\{1,\dots,i\}}\right) : w\in\mathfrak{S}^\alpha \right\}.$$

Recall that $x_I := \prod_{i\in I} x_1\cdots x_i$ for all $I\subseteq[n]$. By sending the above basis to the set $\left\{\overline{\pi}_w x_{D(\alpha)} : w\in\mathfrak{S}^\alpha\right\}$ which is triangularly related to $\left\{w x_{D(\alpha)} : w\in\mathfrak{S}^\alpha\right\}$ (see the previous chapter), one has the desired isomorphism.

4.3.2 Tits Building

Let $\Delta(G)$ be the Tits building of the general linear group $G=GL(n,\mathbb{F}_q)$ and its usual BN-pair over a finite field $\mathbb{F}_q$; see e.g. Björner [9]. The Stanley-Reisner ring $\mathbb{F}[\Delta(G)]$ is a q-analogue of $\mathbb{F}[\mathcal{B}_n]$. The nonzero monomials in $\mathbb{F}[\Delta(G)]$ are indexed by multiflags of subspaces of $\mathbb{F}_q^n$, and there are $q^{\mathrm{inv}(w)}$ many multiflags corresponding to a given multichain M in $\mathcal{B}_n$, where $w = \sigma(M)$. Can one obtain the multivariate quasisymmetric function identities in Theorem 4.2.14 by defining a nice $H_n(0)$-action on $\mathbb{F}[\Delta(G)]$?

Chapter 5

Hecke algebra action on the Stanley-Reisner ring of the Coxeter complex

In the previous chapter we defined an action of the 0-Hecke algebra $H_n(0)$ of $\mathfrak{S}_n$ on the Stanley-Reisner ring of the Boolean algebra. In this chapter we generalize it in two directions, $0 \to q$ and $\mathfrak{S}_n \to W$, by defining an action of the Hecke algebra $H_W(q)$ of a finite Coxeter group W on the Stanley-Reisner ring of the Coxeter complex of W.

5.1 Stanley-Reisner ring of the Coxeter complex

In §2.2 we provided the definitions for the Stanley-Reisner ring of a simplicial complex and the Coxeter complex $\Delta(W)$ of a finite Coxeter system (W,S). See Björner [9] and Garsia-Stanton [27] for more background information.

The Stanley-Reisner ring $\mathbb{F}[\Delta(W)]$ has an $\mathbb{F}$-basis of all nonzero monomials. If $m = v_1 \cdots v_k$ is a nonzero monomial, then $\mathrm{supp}(m) = wW_J$ for some $w \in W^J$, where J is the underlying set of the rank multiset $r(m)$. There is a natural W-action on $\mathbb{F}[\Delta(W)]$ by $w(m) := w(v_1) \cdots w(v_k)$ for all $w \in W$. This action preserves the multigrading of $\mathbb{F}[\Delta(W)]$, and is transitive on every homogeneous component of $\mathbb{F}[\Delta(W)]$. Let Θ be the set of the rank polynomials

$$\theta_i = \sum_{w \in W^{i^c}} wW_{i^c}, \quad i = 1, \ldots, d.$$

The W-action is Θ-linear and leaves the polynomial algebra $\mathbb{F}[\Theta]$ invariant.

Proposition 5.1.1 (c.f. Garsia and Stanton [27])**.** *The invariant algebra* $\mathbb{F}[\Delta]^W$ *equals* $\mathbb{F}[\Theta]$.

Proof. It suffices to show $\mathbb{F}[\Delta(W)]^W \subseteq \mathbb{F}[\Theta]$. The W-action on $\mathbb{F}[\Delta(W)]$ breaks up the set of nonzero monomials into orbits, and the orbit sums form an $\mathbb{F}$-basis for $\mathbb{F}[\Delta(W)]^W$. The W-orbit of a nonzero monomial with rank multiset $\{1^{a_1}, \ldots, d^{a_d}\}$ consists of all monomials with the same rank multiset, and hence the corresponding orbit sum equals $\theta_1^{a_1} \cdots \theta_d^{a_d}$. □

Let $J \subseteq S$. The rank-selected subcomplex $\Delta_J(W)$ has chambers wW_{J^c} for all w in W^{J^c}. A shelling order is obtained from any linear extension of the weak order on W^{J^c}, and the restriction map is given by $R_J(wW_{J^c}) = wW_{D(w)^c}$ for all w in W^{J^c}. The W-action on $\Delta(W)$ restricts to $\Delta_J(W)$, inducing a W-action on the Stanley-Reisner ring $\mathbb{F}[\Delta_J(W)]$. Let $\Theta_J := \{\theta_j : j \in J\}$. Then Theorem 2.2.2 implies that $\mathbb{F}[\Delta_J(W)]$ is a free $\mathbb{F}[\Theta_J]$-module with a basis of the descent monomials

$$wW_{D(w)^c} = \prod_{i \in D(w)} wW_{i^c}, \quad \forall w \in W^{J^c}.$$

The W-action on $\mathbb{F}[\Delta_J(W)]$ is Θ_J-linear, and thus descends to the quotient algebra $\mathbb{F}[\Delta_J(W)]/(\Theta_J)$.

5.2 Hecke algebra action

Suppose that $\mathbb{F}$ is an arbitrary field, q is an indeterminate, and (W, S) is a finite Coxeter system with $S = \{s_1, \ldots, s_d\}$. We define an action of the Hecke algebra of W on the Stanley-Reisner ring $\mathbb{F}(q)[\Delta(W)]$ of the Coxeter complex $\Delta(W)$ of W, so that $\mathbb{F}(q)[\Delta(W)]$ becomes a multigraded $H_W(q)$-module.

First recall from §2.4 that for every $J \subseteq S$, the parabolic subalgebra $H_{W,J}(q)$ is generated by $\{T_j : s_j \in J\}$, and acts trivially on the element

$$\sigma_J := \sum_{w\in W_J} T_w.$$

by

$$T_w\sigma_J = q^{\ell(w)}\sigma_J, \quad \forall w \in W_J.$$

The induction of $H_{W,J}(q)\sigma_J$ to $H_W(q)$ gives the parabolic representation $H_W(q)\sigma_J$, which has an $\mathbb{F}(q)$-basis $\{T_w\sigma_J : w \in W^J\}$.

Let m be a nonzero monomial in $\mathbb{F}(q)[\Delta(W)]$ with $\mathrm{supp}(m) = wW_J$, where $w \in W^J$. We define

$$T_i(m) := \begin{cases} (q-1)m + qs_i(m), & \text{if } i \in D(w^{-1}), \\ qm, & \text{if } i \notin D(w^{-1}),\ s_iw \notin W^J, \\ s_i(m), & \text{if } i \notin D(w^{-1}),\ s_iw \in W^J. \end{cases} \tag{5.1}$$

This gives an $H_W(q)$-action on $\mathbb{F}(q)[\Delta(W)]$ preserving the multigrading, according to the following result.

Proposition 5.2.1. *Let I be a multiset with underlying set $J^c \subseteq [d]$. Then the homogeneous component of $\mathbb{F}(q)[\Delta(W)]$ indexed by I is isomorphic to the parabolic representation $H_W(q)\sigma_J$.*

Proof. Recall that the parabolic representation $H_W(q)\sigma_J$ has a basis $\{T_w\sigma_J : w \in W^J\}$. On the other hand, the homogeneous component of $\mathbb{F}(q)[\Delta(W)]$ indexed by I has a natural basis of all nonzero monomials m with $r(m) = I$, and any such monomial m has $\mathrm{supp}(m) = wW_J$ for some $w \in W^J$, since J^c is the underlying set of I. Thus we obtain a vector space isomorphism by sending m to $T_{\sigma(m)}\sigma_J$ for all nonzero monomials m with $r(m) = \bar{J}$. This isomorphism is $H_W(q)$-equivariant by the following observations.

If $i \in D(w^{-1})$, then $D(s_iw) \subseteq D(w) \subseteq J^c$ and $T_iT_w\sigma_J = ((q-1)T_w + qT_{s_iw})\sigma_J$.

If $i \notin D(w^{-1})$ and $s_iw \notin W^J$, then $s_iw = ws_j$ for some $j \in J$ by Lemma 2.1.1, and thus $T_iT_w\sigma_J = T_wT_j\sigma_J = qT_w\sigma_J$.

If $i \notin D(w^{-1})$ and $s_iw \in W^J$, then $T_iT_w\sigma_J = T_{s_iw}\sigma_J$. $\square$

Specializing $q = 0$ we get an $H_W(0)$-action on $\mathbb{F}[\Delta(W)]$ by the operators $\overline{\pi}_i := T_i|_{q=0}$. One directly verifies $T_i = qs_i + (1-q)\overline{\pi}_i$ for all $i \in [d]$.

Proposition 5.2.2. *The actions of* W *and* $H_W(0)$ *on* $\mathbb{F}[\Delta(W)]$ *satisfy the relations*

$$\overline{\pi}_i s_i = -\overline{\pi}_i, \quad s_i \overline{\pi}_i = \overline{\pi}_i + 1 - s_i, \quad 1 \le i \le n-1.$$

Proof. Using $T_i = qs_i + (1-q)\overline{\pi}_i$ one sees that $(T_i + 1)(T_i - q) = 0$ holds if and only if

$$(q^2 - q)(\overline{\pi}_i s_i + s_i \overline{\pi}_i + s_i - 1) = 0.$$

Here q is an indeterminate. Thus it suffices to show $\overline{\pi}_i s_i = -\overline{\pi}_i$. Let m be a nonzero monomial in $\mathbb{F}[\Delta(W)]$ with $\mathrm{supp}(m) = wW_J$ where $J \subseteq S$ and $w \in W^J$.

If $i \in D(w^{-1})$ then $\overline{\pi}_i(m) = -m$ and $\mathrm{supp}(s_i(m)) = s_i w W_J$ with $s_i w \in W^J$. Thus

$$\overline{\pi}_i s_i(m) = s_i^2(m) = m = -\overline{\pi}_i(m).$$

If $i \notin D(w^{-1})$ and $s_i w \notin W^J$, then $\overline{\pi}_i(m) = 0$ and $s_i(m) = m$. Hence

$$\overline{\pi}_i s_i(m) = 0 = -\overline{\pi}_i(m).$$

If $i \notin D(w^{-1})$ and $s_i w \in W^J$, then $\overline{\pi}_i(m) = s_i(m)$ and thus

$$\overline{\pi}_i s_i(m) = \overline{\pi}_i^2(m) = -\overline{\pi}_i(m).$$

The proof is complete. □

5.3 Invariants and coinvariants

The *trivial representation* of $H_W(q)$ is a one-dimensional $\mathbb{F}(q)$-space on which T_w acts by $q^{\ell(w)}$ for all $w \in W$; it is a well defined $H_W(q)$-module by the defining relations for $H_W(q)$. We define the *invariant algebra* of the $H_W(q)$-action on $\mathbb{F}(q)[\Delta(W)]$ to be the trivial isotypic component

$$\mathbb{F}(q)[\Delta(W)]^{H_W(q)} := \{f(q) \in \mathbb{F}[\Delta(W)] : T_i(f) = qf, \ \forall i \in [d]\}.$$

Although not a priori obvious, we show below that it is indeed an algebra.

Proposition 5.3.1. *If* q *is an indeterminate or* $q \in \mathbb{F}$ *then* $\mathbb{F}(q)[\Delta(W)]^{H_W(q)} = \mathbb{F}(q)[\Theta]$.

Proof. For $j = 1, 2, 3$ let $\mathcal{M}_j$ be the set of nonzero monomials in the j-the case of the definition (5.1) of the $H_W(q)$-action. Let $i \in [d]$. Then s_i pointwise fixes $\mathcal{M}_2$ and bijectively sends $\mathcal{M}_1$ to $\mathcal{M}_3$. Hence every element in $\mathbb{F}(q)[\Delta(W)]$ can be written as

$$f = \sum_{m \in \mathcal{M}_1} (c_m m + c_{s_i m} s_i m) + \sum_{m \in \mathcal{M}_2} c_m m.$$

It follows that

$$T_i(f) - qf = \sum_{m \in \mathcal{M}_1} \left((c_{s_i m} - c_m)m + q(c_m - c_{s_i m})s_i m\right).$$

This shows that $T_i(f) = qf$ if and only if $c_{s_i m} = c_m$ for all $m \in \mathcal{M}_1$ if and only if $s_i(f) = f$. Hence $\mathbb{F}(q)[\Delta(W)]^{H_W(q)} = \mathbb{F}(q)[\Delta(W)]^W = \mathbb{F}(q)[\Theta]$ by Proposition 5.1.1. $\square$

Remark 5.3.2. One has another proof by applying s_i to $T_i(f) = qf$ and using the relation $s_i\overline{\pi}_i = \overline{\pi}_i + 1 - s_i$ to get $s_i(f) = f$, as long as $q \neq -1$.

Thus the quotient algebra $\mathbb{F}(q)[\Delta(W)]/(\Theta)$ can be viewed as the coinvariant algebra of the $H_W(q)$-action on $\mathbb{F}(q)[\Delta(W)]$. It inherits the multigrading of $\mathbb{F}(q)[\Delta(W)]$ since the ideal (Θ) is homogeneous. We prove below that the operators T_i are Θ-linear, and thus apply to $\mathbb{F}(q)[\Delta(W)]/(\Theta)$.

Proposition 5.3.3. *The $H_W(q)$-action on $\mathbb{F}(q)[\Delta(W)]$ is Θ-linear if q is an indeterminate or $q \in \mathbb{F}$.*

Proof. Fix an $i \in [d]$. Since $T_i = qs_i + (1-q)\overline{\pi}_i$ and s_i is Θ-linear, it suffices to show that $\overline{\pi}_i$ is Θ-linear. Let m be a nonzero monomial in $\mathbb{F}[\Delta(W)]$ with $\mathrm{supp}(m) = wW_J$, where $J \subseteq S$ and $w \in W^J$. We need to show $\overline{\pi}_i(\theta_j m) = \theta_j \overline{\pi}_i(m)$ for all j. It follows from the definition of θ_j that

$$\theta_j m = \sum_{u \in W(j,m)} uW_{j^c} \cdot m$$

where $W(j,m) := \{u \in W^{j^c} : uW_{j^c} \cap wW_J \neq \emptyset\}$. For every $u \in W(j,m)$, let z be the shortest element in $uW_{j^c} \cap wW_J$. Then one has

$$\mathrm{supp}(uW_{j^c} \cdot m) = uW_{j^c} \cap wW_J = zW_{J \setminus \{j\}}, \quad z \in W^{J \setminus \{j\}}. \tag{5.2}$$

By definition, the action of T_i on $uW_{j^c} \cdot m$ depends on the following three cases:

(i) $i \in D(z^{-1})$,

(ii) $i \notin D(z^{-1})$ and $D(s_i z) \not\subseteq J^c \cup \{j\}$,

(iii) $i \notin D(z^{-1})$ and $D(s_i z) \subseteq J^c \cup \{j\}$.

This decomposes $W(j,m)$ into a disjoint union of three subsets $W_1(j,m)$, $W_2(j,m)$, and $W_3(j,m)$.

Now we distinguish three cases below.

Case 1: $i \in D(w^{-1})$. Then $\overline{\pi}_i(m) = -m$. Let $u \in W(j,m)$ and assume (5.2). Since $z \in wW_J$ and $w \in W^J$, we have $D(w^{-1}) \subseteq D(z^{-1})$. Thus $W(j,m) = W_1(j,m)$ and $\overline{\pi}_i(\theta_j m) = -\theta_j m = \theta_j \overline{\pi}_i(m)$.

Case 2: $i \notin D(w^{-1})$ and $s_i w \notin W^J$. Then $s_i w W_J = wW_J$, $\overline{\pi}_i(m) = 0$, and $s_i(m) = m$. Again we let $u \in W(j,m)$ and assume (5.2).

If $u \in W_1(j,m)$, i.e. $i \in D(z^{-1})$, then $D(s_i z) \subseteq D(z) \subseteq J^c \cup \{j\}$ and

$$s_i u W_{j^c} \cap wW_J = s_i(uW_{j^c} \cap wW_J) = s_i z W_{J \setminus \{j\}} \neq \emptyset.$$

Hence $s_i u \in W_3(j,m)$ and $\overline{\pi}_i(s_i u W_{j^c} \cdot m) = s_i(s_i u W_{j^c} \cdot m) = uW_{j^c} \cdot m$.

Similarly, if $u \in W_3(j,m)$ then $s_i u \in W_1(j,m)$ and $\overline{\pi}_i(uW_{j^c} \cdot m) = s_i u W_{j^c} \cdot m$.

If $u \in W_2(j,m)$ then $\overline{\pi}_i(uW_{j^c} \cdot m) = 0$. It follows that

$$\overline{\pi}_i(\theta_j m) = \sum_{u \in W_3(j,m)} \overline{\pi}_i(1 + \overline{\pi}_i)(uW_{j^c} \cdot m) = 0 = \theta_j \overline{\pi}_i(m).$$

Case 3: $i \notin D(w^{-1})$ and $s_i w \in W^J$. Then $\overline{\pi}_i(m) = s_i(m)$. Let $u \in W(j,m)$ and assume (5.2). Then $z = wy$ for some $y \in W_J$. Since $s_i w \in W^J$, one has

$$\ell(s_i z) = \ell(s_i w) + \ell(y) = 1 + \ell(w) + \ell(y) = 1 + \ell(z)$$

which implies $i \notin D(z^{-1})$. If $s_i z \notin W^J$ then $s_i w W_J = s_i z W_J = zW_J = wW_J$ by Lemma 2.1.1, which leads us back to Case 2. Hence one has $W(j,m) = W_3(j,m)$ and $\overline{\pi}_i(\theta_j m) = s_i(\theta_j m) = \theta_j s_i(m) = \theta_j \overline{\pi}_i(m)$. □

Now we know that the coinvariant algebra $\mathbb{F}(q)[\Delta(W)]/(\Theta)$ is a multigraded $H_W(q)$-module.

Theorem 5.3.4. *The coinvariant algebra $\mathbb{F}(q)[\Delta(W)]/(\Theta)$ carries the regular representation of $H_W(q)$ if q is generic, i.e. if q is an indeterminate or $q \in \mathbb{F} \setminus E$ for some finite set $E \subsetneq \mathbb{F}$.*

Proof. It follows from Theorem 5.4.1 (to be proved in the next section) that setting $q = 0$, the coinvariant algebra $\mathbb{F}[\Delta(W)]/(\Theta)$ is isomorphic to the regular representation of $H_W(0)$, i.e. the desired result holds for $q = 0$. Thus there exists an element $f \in \mathbb{F}[\Delta(W)]$ such that $\{\overline{\pi}_w f : w \in W\}$ gives an $\mathbb{F}$-basis for $\mathbb{F}[\Delta(W)]/(\Theta)$. Then it suffices to show that $\{T_w f : w \in W\}$ gives an $\mathbb{F}(q)$-basis for $\mathbb{F}(q)[\Delta(W)]/(\Theta)$.

Let $\mathbb{F}[q][\Delta(W)] := \mathbb{F}[\Delta(W)] \otimes \mathbb{F}[q]$ be the Stanley-Reisner ring of $\Delta(W)$ over the polynomial algebra $\mathbb{F}[q]$. For any $w \in W$, the element $T_w f$ belongs to $\mathbb{F}[q][\Delta(W)]$ and we identify it with its image in $\mathbb{F}[q][\Delta(W)]/(\Theta)$. Then using the descent basis for $\mathbb{F}[\Delta(W)]/(\Theta)$ we obtain

$$T_w f = \sum_{u \in W} a_{wu}(q) \cdot u W_{D(u)^c} \quad \text{inside } \mathbb{F}[q][\Delta(W)]/(\Theta)$$

where $a_{uw}(q) \in \mathbb{F}[q]$ for all pairs of $u, w \in W$. Thus $A(q) := [a_{uw}]_{u,w \in W}$ is the transition matrix between the basis $\{uW_{D(u)^c} : u \in W\}$ of descent monomials and the desired basis $\{T_w f : w \in W\}$ for $\mathbb{F}(q)[\Delta(W)]/(\Theta)$. It remains to show $\det A(q) \neq 0$. But we know $\det A(0) \neq 0$ since $\{\overline{\pi}_w f : w \in W\}$ gives an $\mathbb{F}$-basis for $\mathbb{F}[\Delta(W)]/(\Theta)$. Thus $\det A(q)$ is a nonzero polynomial in $\mathbb{F}[q]$ which has only finitely many roots. It follows that $\det A(q) \neq 0$ if q is generic. $\square$

Remark 5.3.5. In those situations where Mathas' decomposition (2.5) of $H_W(q)$ holds, one has the following constructive proof for Theorem 5.3.4. By Proposition 5.2.1, the cyclic $H_W(q)$-module generated by

$$W_J = \prod_{i \in J^c} W_{i^c} \quad \text{inside } \mathbb{F}(q)[\Delta(W)]$$

is isomorphic to $H_W(q)\sigma_J$. This induces an $H_W(q)$-homomorphism

$$\psi : H_W(q) = \bigoplus_{J \subseteq S} \Pi_J \hookrightarrow \mathbb{F}(q)[\Delta(W)] \twoheadrightarrow \mathbb{F}(q)[\Delta(W)]/(\Theta).$$

By (2.4), we have an $\mathbb{F}(q)$-basis for $\psi(\Pi_J)$ given by

$$\left\{ y_u(q) = \sum_{w \in W_{J^c}} (-q)^{\ell(w)} T_{uw} W_J : u \in W,\ D(u) = J^c \right\}.$$

Note that $\{y_u(0) = uW_{D(u)^c} : u \in W\}$ is precisely the descent basis for $\mathbb{F}[\Delta(W)]/(\Theta)$. By the same technique used in the above proof of Theorem 5.3.4, we can show that if q is generic then $\{y_u(q) : u \in W\}$ gives an $\mathbb{F}(q)$-basis for $\mathbb{F}(q)[\Delta(W)]/(\Theta)$ and thus ψ is an isomorphism of $H_W(q)$-modules.

5.4 0-Hecke algebra action

Now we study the action of the 0-Hecke algebra $H_W(0)$ on the Stanley-Reisner ring $\mathbb{F}[\Delta(W)]$ by the operators $\overline{\pi}_i := T_i|_{q=0}$. Since this action preserves the rank of the faces of $\Delta(W)$, it restricts to the Stanley-Reisner ring $\mathbb{F}[\Delta_J(W)]$ of the rank-selected subcomplex $\Delta_J(W)$ of $\Delta(W)$, giving $\mathbb{F}[\Delta_J(W)]$ a multigraded $H_W(0)$-module structure, for all $J \subseteq S$.

Theorem 5.4.1. *Let $J \subseteq S$ and $\Theta_J := \{\theta_j : j \in J\}$. Then we have an $H_W(0)$-module decomposition*

$$\mathbb{F}[\Delta_J(W)]/(\Theta_J) = \bigoplus_{I \subseteq J} N_I$$

where N_I is the $\mathbb{F}$-span of $\{wW_{I^c} : D(w) = I\}$ inside $\mathbb{F}[\Delta_J(W)]/(\Theta_J)$ for all $I \subseteq J$. Moreover, each N_I has homogeneous multigrading $\underline{t}^I$ and is isomorphic to the projective indecomposable $H_W(0)$-module $\mathbf{P}_I$. In particular, we have an $H_W(0)$-module isomorphism $\mathbb{F}[\Delta(W)]/(\Theta) \cong H_W(0)$ for any field $\mathbb{F}$.

Proof. By Theorem 2.2.2, $\{wW_{D(w)^c} : D(w) \subseteq J\}$ gives an $\mathbb{F}$-basis for $\mathbb{F}[\Delta_J(W)]/(\Theta_J)$. Therefore $\mathbb{F}[\Delta_J(W)]/(\Theta_J)$ is a direct sum of N_I for all $I \subseteq J$ as an $\mathbb{F}$-vector space. Each N_I is homogeneous with multigrading $\underline{t}^I$.

Taking $q = 0$ in Proposition 5.2.1 one has an $H_W(0)$-module isomorphism from the homogeneous component of $\mathbb{F}[\Delta(W)]$ indexed by I to the parabolic representation $H_W(0)\pi_{w_0(I^c)}$ by sending wW_{I^c} to $\overline{\pi}_w \pi_{w_0(I^c)}$ for all $w \in W$ with $D(w) \subseteq I$. This induces the desired isomorphism $N_I \cong \mathbf{P}_I$ of $H_W(0)$-modules.

Finally in the special case of $J = S$ one has an $H_W(0)$-module decomposition of $\mathbb{F}[\Delta(W)]/(\Theta)$ which agrees with Norton's decomposition of $H_W(0)$. Hence $\mathbb{F}[\Delta(W)]/(\Theta)$ carries the regular representation of $H_W(0)$. $\square$

5.5 Questions for future research

5.5.1 For which q is the coinvariant algebra regular?

As shown in Theorem 5.3.4, the coinvariant algebra $\mathbb{F}(q)[\Delta(W)]/(\Theta)$ carries the regular $H_W(q)$-representation if q generic. It is interesting to study the following problem.

Problem. *Determine the finitely many values of q for which $\mathbb{F}(q)[\Delta(W)]/(\Theta) \ncong H_W(q)$.*

For instance, Theorem 5.4.1 implies the isomorphism $\mathbb{F}[\Delta(W)]/(\Theta) \cong H_W(0)$ for an arbitrary field $\mathbb{F}$.

Another example is for $W = \mathfrak{S}_n$ and $q = 1$. By the discussion given in the end of last chapter one has

$$\mathrm{ch}_t(\mathbb{C}[\mathcal{B}_n]/(\Theta)) = \mathrm{ch}_t(\mathbb{C}[X]/(\mathbb{C}[X]_+^{\mathfrak{S}_n}))$$

where the first t is obtained from the specialization $t_i = t^i$ for the $\mathbb{N}^n$-multigrading of $\mathbb{C}[\mathcal{B}_n]$, and the second t keeps track of the degree grading of $\mathbb{C}[X]$.

This implies an isomorphism $\mathbb{C}[\mathcal{B}_n]/(\Theta) \cong \mathbb{C}[X]/(\mathbb{C}[X]_+^{\mathfrak{S}_n})$ of graded $\mathfrak{S}_n$-modules, which shows that $\mathbb{C}[\mathcal{B}_n]/(\Theta)$ carries the regular representation of $\mathfrak{S}_n$. However, we do not know any explicit construction of this isomorphism; in particular, it is not induced by the transfer map τ, as τ does not send the ideal (Θ) to the ideal $(\mathbb{C}[X]_+^{\mathfrak{S}_n})$ (e.g. $\tau(\theta_1\{1\}) = x_1^2 \notin (\mathbb{C}[X]_+^{\mathfrak{S}_n})$ when $n \geq 3$).

On the other hand, one sees that $\mathbb{F}[\mathcal{B}_2]/(\theta_1)$ is a direct sum of the trivial representation and the sign representation, and hence isomorphic to the regular representation of $\mathfrak{S}_2$ if and only if $2 \nmid \mathrm{char}(\mathbb{F})$.

5.5.2 Character formula

Adin, Postnikov, and Roichman [2] studied the $H_n(q)$-action (via Demazure operators) on $\mathbb{C}(q)[X]/(\mathbb{C}(q)[X]_+^{\mathfrak{S}_n})$ and found a character formula for the k-th homogeneous component using the basis of Schubert polynomials; the result is a sum of certain q-weights of all permutations in $\mathfrak{S}_n$ of length k.

Now we have an $H_n(q)$-action on $\mathbb{C}(q)[\mathcal{B}_n]/(\Theta)$. We do not know any analogue of the Schubert polynomials in $\mathbb{C}(q)[\mathcal{B}_n]/(\Theta)$, but the descent basis for $\mathbb{C}(q)[\mathcal{B}_n]/(\Theta)$ behaves nicely under the $H_n(q)$-action. Therefore we propose the following problem.

Problem. *Use the descent basis to find a character formula for the homogeneous components of the $H_n(q)$-module $\mathbb{C}(q)[\mathcal{B}_n]/(\Theta)$, expressing it as a sum of some q-weights of all permutations in $\mathfrak{S}_n$ with a fixed major index.*

More generally, given a finite Coxeter group W generated by S, it remains open to study the multigraded $H_W(q)$-module structure on $\mathbb{F}(q)[\Delta_J(W)]/(\Theta_J)$ for all $J \subseteq S$. We only solved the specialization at $q = 0$ (see Theorem 5.4.1).

5.5.3 Gluing the group algebra and the 0-Hecke algebra

The group algebra $\mathbb{F}W$ of a finite Coxeter group W naturally admits both actions of W and $H_W(0)$. Hivert and Thiéry [32] defined the *Hecke group algebra* of W by gluing these two actions. In type A, one can also glue the usual actions of $\mathfrak{S}_n$ and $H_n(0)$ on the polynomial ring $\mathbb{F}[X]$, but the resulting algebra is different from the Hecke group algebra of $\mathfrak{S}_n$.

Now one has a W-action and an $H_W(0)$-action on the Stanley-Reisner ring $\mathbb{F}[\Delta(W)]$. What can we say about the algebra generated by the operators s_i and $\overline{\pi}_i$ on $\mathbb{F}[\Delta(W)]$? Is it the same as the Hecke group algebra of W? If not, what properties (dimension, bases, presentation, simple and projective indecomposable modules, etc.) does it have?

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