Nilpotent ideals of upper triangular matrices and Variations of the Catalan numbers

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Definition

- Let $\mathcal{U}_n$ be the algebra of all n-by-n upper triangular matrices

$$\begin{pmatrix} * & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ 0 & 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & * \end{pmatrix}$$

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- A ideal I of $\mathcal{U}_n$ is commutative if $AB = BA$ for all $A, B \in I$.

Nilpotent ideals

Example (A nilpotent ideal of $\mathcal{U}_6$ and its corresponding Dyck path)

$$I = \begin{bmatrix} 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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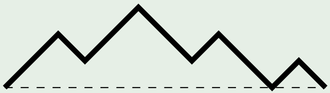

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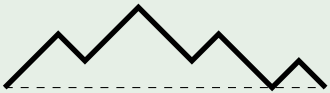

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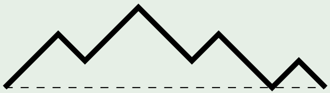

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- *The number of all ideals of $\mathcal{U}_n$ is the Catalan number C_{n+1}.*

Commutative ideals

Proposition (L. Shapiro, 1975)

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The number of subsets of $\{1, 2, \ldots, n\}$ is $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$.
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Let C_n^d be the number of nilpotent ideals of $\mathcal{U}_n$ with order at most d.

Nilpotent order

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The order of a nilpotent ideal I of $\mathcal{U}_n$ is the largest possible length d of an admissible sequence, that is, a sequence $(i_1, i_2, \ldots, i_d)$ such that the entry (i_j, i_{j+1}) is a star $*$ in the matrix form of I for all $j = 1, 2, \ldots, d-1$.

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Example

The following ideal has nilpotent order is 4 since the sequence $(1, 3, 5, 6)$ is admissible and there is no longer admissible sequence.

$$I = \begin{bmatrix} 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Bounce Paths

Observation

Let I be an ideal of $\mathcal{U}_n$ corresponding to a Dyck path D. Then the nilpotent order of I is the number of times the bounce path of D bounces off the main diagonal.

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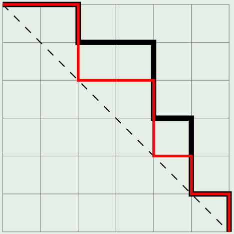

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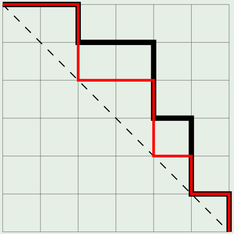

- The bounce path has 4 bounces.
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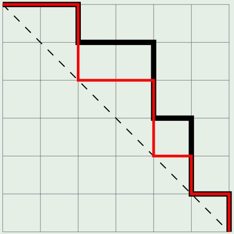

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Fact (Andrews–Krattenthaler–Orsina–Papi 2002, Haglund 2008)

Bijection ζ : Dyck paths with height $d \leftrightarrow$ Dyck paths with d bounces.

Generalization of Commutative Ideals

Theorem (H.-Rhoades)

Dyck paths of length $2n$ with height at most d are counted by C_n^d. Hence C_n^d is the sequence A080934 in OEIS and interpolates between 1 and C_n.

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Example

n	1	2	3	4	5	6	7	n
C_n^1	1	1	1	1	1	1	1	1
C_n^2	1	2	4	8	16	32	64	2^{n-1}
C_n^3	1	2	5	13	34	89	233	F_{2n-1}
C_n^4	1	2	5	14	41	122	365	$\frac{1}{2}(1+3^{n-1})$
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Problem

Is there a nice (q,t)-analogue of the number C_n^d?

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The number of ad-nilpotent ideals of $\mathfrak{b}$ with order at most $d-1$ is C_n^d. This gives a nonsymmetric (q, t)-analogue of C_n using (#bounces, area).

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- *The above theorem has been generalized from type A to other types [Krattenthaler–Orsina–Papi 2002]. Is there a similar generalization for nilpotent ideals of $\mathcal{U}_n$?*

Generating function

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Example

$C^1(x) = \frac{x}{1-x}$, $C^2(x) = \frac{x}{1-\frac{x}{1-x}} = \frac{x(1-x)}{1-2x}$, $C^3(x) = \frac{x}{1-\frac{x}{1-\frac{x}{1-x}}} = \frac{x(1-2x)}{1-3x+x^2}$

Closed Formulas for C_n^d

Theorem (Andrews–Krattenthaler–Orsina–Papi 2002)

The number C_n^d has the following closed formulas:

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$$= \sum_{0=i_0\le i_1\le\cdots\le i_{d-1}\le i_d=n}\ \prod_{0\le j\le d-2}\binom{i_{j+2}-i_j-1}{i_{j+1}-i_j}.$$

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Theorem (de Bruijn–Knuth–Rice 1972)

The number of plane trees with $n+1$ nodes of depth at most d is

$$C_n^d = \frac{2^{2n+1}}{d+2} \sum_{1 \le j \le d+1} \sin^2(j\pi/(d+2)) \cos^{2n}(j\pi/(d+2)).$$

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A four-parameter generalization of C_n^d

Definition

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Binary trees

Fact

Parenthesizations of $z_0 \oplus \cdots \oplus z_n \leftrightarrow$ *(full) binary trees with* $n+1$ *leaves.*

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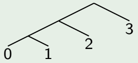

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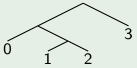

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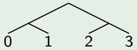

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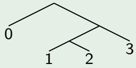

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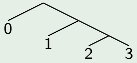

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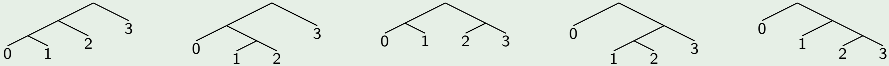

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A parenthesization of $z_0 \oplus \cdots \oplus z_n$ corresponding to $t \in \mathcal{T}_n$ equals

$$x^{\delta_0(t)} y^{\rho_0(t)} z_0 + \cdots + x^{\delta_n(t)} y^{\rho_n(t)} z_n$$

where the left depth $\delta_i(t)$ (or right depth $\rho_i(t)$, resp.) of leaf i in $t \in \mathcal{T}_n$ is the number of edges to the left (or right, resp.) in the unique path from the root of t down to i.

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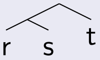
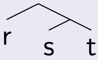
- The rotation for binary trees is given by 

$\leftrightarrow$ 

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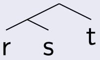
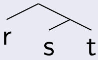
- The rotation for binary trees is given by 

$\leftrightarrow$ 

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If $e = k = \ell = 1$ then two parenthesizations of $z_0 \oplus \cdots \oplus z_n$ are equal if and only if their corresponding binary trees can be obtained from each other by applying rotations at nodes of left depth at least $d - 1$.

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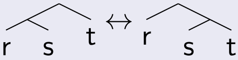
- The rotation for binary trees is given by 

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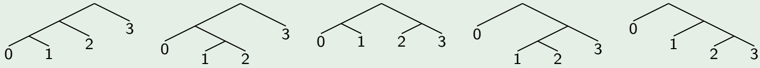
Example

The first two binary trees are equivalent for $\oplus$ with $d = 2$, $e = k = \ell = 1$.



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For $n, d \geq 1$ we have

$$C_n^d = \sum_{\substack{\alpha \models n \\ \max(\alpha) \leq (d+1)/2}} (-1)^{n-\ell(\alpha)} \binom{d-\alpha_1}{\alpha_1 - 1} \prod_{2 \leq i \leq \ell(\alpha)} \binom{d+1-\alpha_i}{\alpha_i}$$

Closed Formulas for C_n^d

Theorem (Andrews–Krattenthaler–Orsina–Papi 2002)

The number C_n^d has the following closed formulas:

$$C_n^d = \sum_{i\in\mathbb{Z}} \frac{2i(d+2)+1}{2n+1}\binom{2n+1}{n-i(d+2)} = \det\left[\binom{i-j+d}{j-i+1}\right]_{i,j=1}^{n-1}$$

$$= \sum_{0=i_0\le i_1\le\cdots\le i_{d-1}\le i_d=n}\ \prod_{0\le j\le d-2}\binom{i_{j+2}-i_j-1}{i_{j+1}-i_j}.$$

Theorem (de Bruijn–Knuth–Rice 1972)

The number of plane trees with $n+1$ nodes of depth at most d is

$$C_n^d = \frac{2^{2n+1}}{d+2}\sum_{1\le j\le d+1}\sin^2(j\pi/(d+2))\cos^{2n}(j\pi/(d+2)).$$

Associativity at left depth d and right depth e

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- *The number* $C_n^{d,e} := C_{1,1,n}^{d,e}$ *enumerates equivalence classes of binary trees with* $n+1$ *leaves under rotations at left depth at least* $d-1$ *and right depth at least* $e-1$.

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Let $C^{d,e}(x) := \sum_{n\geq 0} C_n^{d,e} x^{n+1}$ for $d, e \geq 1$. Let $C^{0,e}(x) := C^{1,e}(x)$ and $C^{d,0}(x) := C^{d,1}(x)$.

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$$C^{d,2}(x) = C^d(x) + \frac{x^{d+2}}{(1-2x)F_{d+2}(x)} \quad \textit{and}$$

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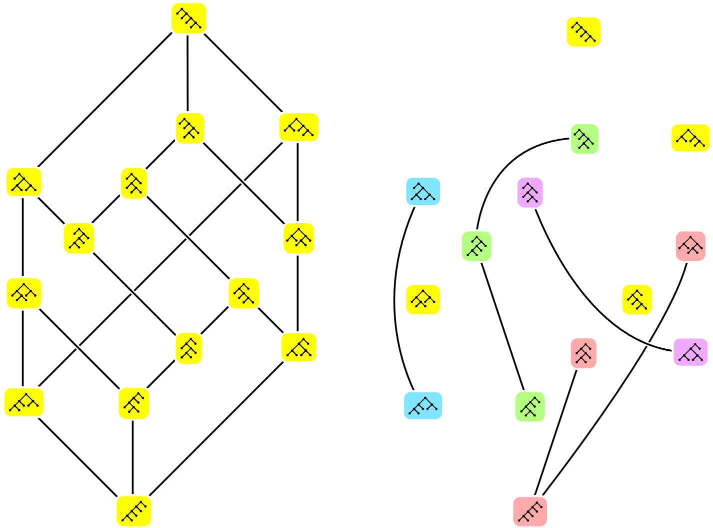
Remark

We have two closed formulas for $C_{k,n}$ and several restricted families of Catalan objects enumerated by $C_{k,n}$ [Hein and H. 2007].

Rotation and 2-rotation



The k-associativity at left depth d

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Let $C^d_{k,n} := C^{d,1}_{k,1,n}$ and $C^d_k(x) := \sum_{n\geq 0} C^d_{k,n} x^{n+1}$.

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If $e = \ell = 1$ then two parenthesizations of $z_0 \oplus \cdots \oplus z_n$ are equal if and only if their corresponding binary trees can be obtained from each other by applying k-rotations at nodes of left depth at least $d-1$.

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The special case $k = 3$

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For $d, n \geq 0$ the number $C^d_{3,n}$ enumerates (a) permutations of $1, 2, \ldots, n$ avoiding 321 and $(d+3)\bar{1}(d+4)2\cdots(d+2)$ [Barcucci–Del Lungo–Pergola–Pinzani 2000], and (b) certain lattice paths [Flajolet 1980].

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$$C_3^d(x) = \frac{2xF_{d+1}(x)F_{d+2}(x) - x^d - x^{d+1} + x^d\sqrt{1-2x-3x^2}}{2(F_{d+2}(x)^2 - x^d - x^{d+1})} \quad \textit{and}$$

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$$C_{3,n}^d = \sum_{\substack{\alpha \models n+1 \\ h>1 \Rightarrow \alpha_h \leq d+1}} \left(-\frac{\delta_{\alpha_1,d}}{2} + (-1)^{\alpha_1 - 1} \sum_{i+j=\alpha_1-1} \binom{d-i}{i}\binom{d+1-j}{j} + \sum_{i+j=\alpha_1-d} \frac{(-3)^i}{2}\binom{\frac{1}{2}}{i}\binom{\frac{1}{2}}{j} \right)$$

$$\cdot \prod_{h\geq 2} \left(\left(\delta_{\alpha_h,d} + (-1)^{\alpha_h - 1} \sum_{i+j=\alpha_h} \binom{d+1-i}{i}\binom{d+1-j}{j} \right) \right)$$

where $\delta_{m,d} := 1$ if $m \in \{d, d+1\}$ or $\delta_{m,d} := 0$ otherwise.

The special case $d = 2$

Theorem (Hein and H.)

For $n \geq 0$ and $k \geq 1$ we have

$$C_{k,n}^2(x) = 1 + \sum_{1 \leq i \leq n-1} \frac{i}{n-i} \sum_{0 \leq j \leq (n-i-1)/k} (-1)^j \binom{n-i}{j} \binom{2n-i-jk-1}{n}$$

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Proposition (Hein and H.)

For $n \geq 0$ we have

$$C^2_{2,n} = \sum_{0 \leq j \leq n} \binom{n+j-1}{2j} = F_{2n-1} \left(= C^3_{1,n} = C^3_n\right).$$

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Conjecture

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- *Find* (q, t)*-analogues of the number* $C^{d,e}_{k,\ell,n}$.

Thank you!