Modular Catalan Numbers

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Outline

- Introduce the *modular Catalan numbers* from parenthesizations
- Use (full) binary trees to study the modular Catalan numbers.
- Give connections between the modular Catalan number and other Catalan objects.
- Investigate closed formulas and generating functions of the modular Catalan numbers.
- Questions for future study.
Binary Operations and Parentheses

Definition

A binary operation \( * \) defined on a set \( X \) is a function \( X \times X \to X \) sending \( (a, b) \in X \times X \) to \( a * b \in X \). (Examples: + and −.)

Observation

The expression \( a * b * c \) is ambiguous and depends on parentheses: \( (a * b) * c \) and \( a * (b * c) \) are not equal in general.

Example (Addition and Subtraction)

Addition is associative: \( (a + b) + c = a + (b + c) \).

Subtraction is NOT associative:

\[
(a - b) - c = a - b - c \\
a - (b - c) = a - b + c
\]
Fact

In general, the number of ways to parenthesize \( x_1 \cdot x_2 \cdot \cdots \cdot x_{n+1} \) is the Catalan number \( C_n := \frac{1}{n+1} \binom{2n}{n} \). (1, 1, 2, 5, 14, 42, ...)

Example (Addition)

Since addition is associative, \( x_1 + \cdots + x_{n+1} \) is unambiguous for any \( n \geq 0 \).

Example (Subtraction, \( n = 3 \))

\[
\begin{align*}
((a-b)-c)-d &= a - b - c - d \\
(a-b)-(c-d) &= a - b - c + d \\
(a-(b-c))-d &= a - b + c - d \\
a-((b-c)-d) &= a - b + c + d \\
a-(b-(c-d)) &= a - b + c - d
\end{align*}
\]
Observation

*Given a binary operation \(*\) on a set \(X\), each parentheization of \(x_1 \ast \cdots \ast x_{n+1}\) gives a function from \(X^{n+1}\) to \(X\).*

Fact

- Denote by \(C_{n,*}\) the number of distinct functions obtained from parenthesizations of \(x_1 \ast \cdots \ast x_{n+1}\). Then we have \(1 \leq C_{n,*} \leq C_n\).
- For example, \(C_{n,+} = 1\), and \(C_{n,-} = 2^{n-1}\).

Definition

- Define a binary operation \(a \circ_k b := a + \omega b\) where \(\omega = e^{2\pi i/k}\) is a primitive \(k\)th root of unity. For example, \(\circ_1\) is + and \(\circ_2\) is −.
- The \((k-)\text{modular Catalan number}\) is \(C_{n,k} := C_{n,\circ_k}\).
### Basic Properties

| $n$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | OEIS     |
|-----|----|----|----|----|----|----|----|----|----|----|----|----|---------|
| $C_{n,1}$ | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1      | A000012  |
| $C_{n,2}$ | 1  | 1  | 2  | 4  | 8  | 16 | 32 | 64 | 128| 256| 512| 1024| 2048   | A000079  |
| $C_{n,3}$ | 1  | 1  | 2  | 5  | 13 | 35 | 96 | 267| 750| 2123| 6046| 17303| 49721  | A005773  |
| $C_{n,4}$ | 1  | 1  | 2  | 5  | 14 | 41 | 124| 384| 1210| 3865| 12482| 40677 | 133572 | A159772  |
| $C_{n,5}$ | 1  | 1  | 2  | 5  | 14 | 42 | 131| 420| 1375| 4576| 15431| 52603 | 180957 | new     |
| $C_{n,6}$ | 1  | 1  | 2  | 5  | 14 | 42 | 132| 428| 1420| 4796| 16432| 56966 | 199444 | new     |
| $C_{n,7}$ | 1  | 1  | 2  | 5  | 14 | 42 | 132| 429| 1429| 4851| 16718| 58331 | 205632 | new     |
| $C_{n,8}$ | 1  | 1  | 2  | 5  | 14 | 42 | 132| 429| 1430| 4861| 16784| 58695 | 207452 | new     |
| $C_n$   | 1  | 1  | 2  | 5  | 14 | 42 | 132| 429| 1430| 4862| 16796| 58786 | 208012 | A000108 |

### Fact

- $C_{n,1} = 1$ and $C_{n,2} = 2^{n-1}$ for $n \geq 0$.
- $C_{0,k} = C_{1,k} = 1$ for $k \geq 1$.
- $C_{n,k} = C_{n}$ for $n \leq k$.
- $C_{k+1,k} = C_{k+1} - 1$ for $k \geq 1$.
- $C_{k+2,k} = C_{k+2} - k - 4$ for $k \geq 2$.
- $C_{k+3,k} = C_{k+3} - (k^2 + 11k + 30)/2$ for $k \geq 3$. 
Binary trees

Observation

*Parenthesesizations of* $x_1 \ast \cdots \ast x_{n+1} \leftrightarrow$ *to binary trees with* $n + 1$ *leaves.*

Example

```
<table>
<thead>
<tr>
<th>Parentheses</th>
<th>Tree</th>
<th>(0, 1, 1, 1)</th>
<th>(0, 1, 1, 2)</th>
<th>(0, 1, 2, 1)</th>
<th>(0, 1, 2, 2)</th>
<th>(0, 1, 2, 3)</th>
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<tr>
<td>$x_1 - x_2 - x_3 - x_4$</td>
<td>1 2 3 4</td>
<td></td>
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</tr>
<tr>
<td>$(x_1 - (x_2 - x_3)) - x_4$</td>
<td>1 2 3 4</td>
<td></td>
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<td></td>
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</tr>
<tr>
<td>$x_1 - x_2 + x_3 - x_4$</td>
<td>1 2 3 4</td>
<td></td>
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</table>

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</tbody>
</table>
```

Definition

The *skew depth* $d_i(t)$ of the $i$th leaf in a binary tree $t$ is the number of steps to the right in the unique downward path from the root of $t$ to $i$. 
Observation

A parenthesization of $x_1 \underbrace{k}_k x_2 \underbrace{k}_k \cdots \underbrace{k}_k x_{n+1}$ is completely determined by the skew depth $d(t) = (d_1, \ldots, d_{n+1})$ of the corresponding tree $t$ modulo $k$, since this parenthesization may be written as

$$\omega^{d_1} x_1 + \omega^{d_2} x_2 + \omega^{d_3} x_3 + \cdots + \omega^{d_{n+1}} x_{n+1}.$$ 

Definition

We say two binary trees with $n + 1$ leaves are $k$-equivalent if their skew depths are congruent modulo $k$.

Theorem (Hein and H.)

Each $k$-equivalence class of binary trees contains exactly one tree avoiding $\text{comb}^1_k$. For example, $\text{comb}_4^4$ and $\text{comb}^1_4$ are given below.

```
     /
    /\  
   / \  
  /   \ 
```

```
     /
    /\  
   / \  
  /   \
```
The modular Catalan number $C_{n,k}$ enumerates the following objects:

1. **Binary trees with $n+1$ leaves avoiding $\text{comb}^1_k$ as a subtree,**
2. **Plane trees with $n+1$ nodes whose non-root nodes have degree less than $k,$**
3. **Dyck paths of length $2n$ avoiding $\text{DU}^k$ (a down-step followed immediately by $k$ consecutive up-steps) as a subpath,**
4. **Partitions with $n$ nonnegative parts bounded by the staircase partition $(n-1, n-2, \ldots, 1, 0)$ such that each positive number appears fewer than $k$ times,**
5. **Standard $2 \times n$ Young tableaux whose top row avoids contiguous labels of the form $i, j+1, j+2, \ldots, j+k$ for all $i < j,$ and**
6. **Permutations of $[n]$ avoiding 1-3-2 and $23 \cdots (k+1).$**
**Definition**

A *plane tree* is a rooted tree for which the children of each node are linearly ordered.

**Fact**

*Binary trees with* \( n + 1 \) *leaves correspond to plane trees with* \( n + 1 \) *nodes via Knuth transform (left-child right-sibling representation of plane trees).*

**Example**

\[\text{Diagram showing correspondence between binary trees and plane trees.}\]
Theorem (Hein and H.)

The modular Catalan number $C_{n,k}$ enumerates the following objects:

1. binary trees with $n + 1$ leaves avoiding $\text{comb}_k^1$ as a subtree,
2. plane trees with $n + 1$ nodes whose non-root nodes have degree less than $k$,
3. Dyck paths of length $2n$ avoiding $\text{DU}^k$ (a down-step followed immediately by $k$ consecutive up-steps) as a subpath,
4. partitions with $n$ nonnegative parts bounded by the staircase partition $(n - 1, n - 2, \ldots, 1, 0)$ such that each positive number appears fewer than $k$ times,
5. standard $2 \times n$ Young tableaux whose top row avoids contiguous labels of the form $i, j+1, j+2, \ldots, j+k$ for all $i < j$, and
6. permutations of $[n]$ avoiding 1-3-2 and $23 \cdots (k + 1)1$. 

Definition

A Dyck path of (semi-)length $2n$, which is a diagonal lattice path from $(0, 0)$ to $(2n, 0)$ consisting of $n$ up-steps $U = (1, 1)$ and $n$ down-steps $D = (1, -1)$ such that none of the path is below the $x$-axis.

Fact

Binary trees with $n + 1$ leaves correspond to Dyck paths of length $2n$.

Example

![Dyck Path Example](image)
Theorem (Hein and H.)

The modular Catalan number $C_{n,k}$ enumerates the following objects:

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6. **Permutations of $[n]$ avoiding $132$ and $23\cdots(k+1)1.$**
Partitions

Definition
- A *partition* if a decreasing sequence of nonnegative integers.
- A partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ correspond to a *Young diagram* with $\lambda_i$ boxes on its $i$th row.

Fact
*Dyck paths of length $2n$ $\leftrightarrow$ partitions $(\lambda_1, \ldots, \lambda_n)$ with $\lambda_i \leq n - i$."

Example
- $\lambda = (3, 1, 0, 0)$
Theorem (Hein and H.)

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6. permutations of $[n]$ avoiding 1-3-2 and $23\cdots(k+1)1$.  

Standard Tableaux

Definition

A *standard tableau of shape* $\lambda$ is a filling of the Young diagram of $\lambda$ with $1, 2, \ldots$ such that each row is increasing from left to right and each column is increasing from top to bottom.

Fact

*Dyck paths of length* $2n$ $\leftrightarrow$ $2 \times n$ *standard tableaux.*

Example

```
   1 2 4 7
   3 5 6 8
```
Theorem (Hein and H.)

The modular Catalan number $C_{n,k}$ enumerates the following objects:

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6. permutations of $[n]$ avoiding 1-3-2 and $23 \cdots (k + 1)$.
**Fact**

*Binary trees with* $n + 1$ *leaves* $\leftrightarrow$ *permutations of* $[n] := \{1, 2, \ldots, n\}$ *avoiding* 1-3-2.

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**Example**

The picture below shows a binary tree with internal nodes labeled with [8]; reading these labels gives a permutation 67534821 avoiding 1-3-2.
Theorem (Hein and H.)

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5. standard $2 \times n$ Young tableaux whose top row avoids contiguous labels of the form $i, j+1, j+2, \ldots, j+k$ for all $i < j$, and
6. permutations of $[n]$ avoiding $1\text{-}3\text{-}2$ and $23\cdots(k+1)1$. 
Generalized Motzkin Numbers

For $n \geq 0$ and $k \geq 1$, the \textit{generalized Motzkin number} $M_{n,k}$ enumerates

1. binary trees with $n + 1$ leaves avoiding $\text{comb}_k$ as a subtree,
2. plane trees with $n + 1$ nodes, each having degree less than $k$ [Takácz],
3. Dyck paths of length $2n$ avoiding $U^k$ ($k$ consecutive up-steps),
4. partitions with $n$ parts bounded by $(n - 1, n - 2, \ldots, 1, 0)$ such that each number appears fewer than $k$ times,
5. $2 \times n$ standard Young tableaux avoiding $k$ consecutive numbers in the top row, and
6. permutations of $[n]$ avoiding 1-3-2 and $12 \cdots k$.

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<th>$M_{n,1}$</th>
<th>$M_{n,2}$</th>
<th>$M_{n,3}$</th>
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Modular Catalan Objects

Theorem (Hein and H.)

The modular Catalan number $C_{n,k}$ enumerates the following objects:

1. binary trees with $n + 1$ leaves avoiding $\text{comb}_k^1$ as a subtree,
2. plane trees with $n+1$ nodes whose non-root nodes have degree less than $k$,
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6. permutations of $[n]$ avoiding $1\cdot3\cdot2$ and $2\cdot3\cdot\cdots(k+1)$.
Proposition (Hein and H.)

- **Recurrence:** \( M_{n,k} = \sum_{0 \leq \ell < k} \sum_{n_1 + \cdots + n_\ell = n - \ell} M_{n_1,k} \cdots M_{n_\ell,k} \).

- **Generating function:** \( M_k(x) := \sum_{n \geq 0} M_{n,k} \) satisfies

\[
M_k(x) = 1 + xM_k(x) + x^2 M_k(x)^2 + \cdots + x^{k-1} M_k(x)^{k-1}.
\]

- **Closed formula (applying Lagrange inversion to the above equation):**

\[
M_{n,k} = \frac{1}{n+1} \sum_{|\lambda| = n, \lambda \subseteq (k-1)^{n+1}} m_\lambda(1, \ldots, 1) \\
= \frac{1}{n+1} \sum_{0 \leq j \leq n/k} (-1)^j \binom{n+1}{j} \binom{2n-jk}{n}.
\]
**Proposition (Hein and H.)**

The generating function \( C_k(x) := \sum_{n \geq 0} C_{n,k} \) satisfies

\[
C_k(x) = \frac{1}{1 - xM_k(x)} = \sum_{\ell \geq 0} (xM_k(x))^\ell.
\]

**Theorem (Hein and H.)**

- For \( n, k \geq 1 \) we have

\[
C_{n,k} = \sum_{\lambda \subseteq (k-1)^n, |\lambda| < n} \frac{n - |\lambda|}{n} m_{\lambda}(1, \ldots, 1)
\]

\[
= \sum_{0 \leq j (n-1)/k} \frac{(-1)^j}{n} \binom{n}{j} \binom{2n - jk}{n + 1}.
\]

- We also have \( x(C_k(x) - 1)^k - xC_k(x)^k + C_k(x)^{k-1} - C_k(x)^{k-2} = 0 \).
Combinatorial Proofs

- Let $w = U^{i_0} DU^{i_1} DU^{i_2} \cdots DU^{i_n}$, with $i_0 > 0$, $i_1, \ldots, i_n \geq 0$, and $i_0 + i_1 + \cdots + i_n = n$.
- Rotation: $w^*_{j} := U^{i_0} DU^{i_{j+1}} \cdots DU^{i_n} DU^{i_{1}} \cdots DU^{i_{j}}$.

\[ w = U^2 DDUDDU \quad w^*^1 = U^2 DUDUD \quad w^*^2 = U^2 DDUDU \quad w^*^3 = U^2 DUDDUD \]

\#\{j \in \{0, 1, \ldots, n-1\} : w^*^j \text{ is a Dyck path}\} = i_0.

This rotation implies the first formula for $C_{n,k}$.

- Coloring one copy of $U$ among $U^{i_0}$ by blue and assuming $DU^k$ occurs at least $j$ times, we get $\binom{n}{j} \left(\frac{2n-jk}{n+1}\right)$ many lattice paths.

- Applying the above rotation and using inclusion-exclusion we prove the second formula for $C_{n,k}$.
The positive sum formula for $C_{n,k}$ becomes the following when $k = 2$:

$$2^{n-1} = \sum_{0 \leq i \leq n-1} \binom{n-1}{i}.$$ 

The $C_{n,2} = 2^{n-1}$ binary trees with $n + 1$ leaves avoiding $\text{comb}_2^1$ form a lattice under the Tamari order, which is isomorphic to the Boolean algebra of subsets of $[n]$ ordered by inclusion.

The rank of a binary tree avoiding $\text{comb}_2^1$ equals the number of non-root internal nodes on its right border.
The 3-modular Catalan numbers \( \{ C_{n,3} \} \) count many other objects:

1. directed \( n \)-ominoes in standard position,
2. \( n \)-digit base three numbers whose digits sum to \( n \),
3. permutations of \( [n] \) avoiding 1-3-2 and 123-4,
4. minimax elements in the affine Weyl group of the Lie algebra \( \mathfrak{so}_{2n+1} \).

Our positive sum formula for \( C_{n,3} \) can be simplified to

\[
C_{n,3} = \sum_{0 \leq i \leq n-1} \binom{n-1}{i} \binom{i}{\lfloor i/2 \rfloor}
\]

which was obtained by Gouyou-Beauchamps and Viennot in their study of the objects in (1) and by Panyushev in his study of objects (4).
Questions

- How is the simplified formula for $C_{n,3}$ related to the modular Catalan objects? Is it possible to generalize this formula to all $k \geq 1$?
- Let $T_{n,k} := \{ \text{binary trees with } n + 1 \text{ leaves avoiding comb}^1_k \}$ be a subposet of the Tamari lattice. What can be said about this poset? How is this poset related to the $(n - 1)$-dimensional associahedron?
- Other modular Catalan objects (noncrossing partitions, triangulations of convex polygons, etc.)?
- Other binary operations?
- We know $1 \leq C_{n,*} \leq C_n$, and $1 = C_{n,*}$ if and only if $*$ is associative. When does $C_{n,*} = C_n$ hold?
Thank you!