Parity of Leaf Depths in Binary Trees

Jianbai Xu

University of Nebraska Kearney

Joint work with J. Huang and M. Mickey

ASSOCIATIVITY OF BINARY OPERATIONS

- Binary operation: $+ \quad - \quad \times \quad \div$ ……
- Associative: (A+B)+C=A+(B+C)
- Non-associative: (A-B)-C≠A-(B-C)
- Question#1:How many possible ways to insert parentheses in

$$X_0 - X_1 - X_2 - \ldots\ldots - X_{n-1} - X_n\,?$$

- The answer is Catalan Number: $C_n = \frac{1}{n+1}\binom{2n}{n} = 1,1,2,5,14,42,132,\ldots\ldots$
- Question#2:How many distinct results?
- The answer is $C_{-,n}=2^{n-1}$:$x_0 - x_1 \pm x_3 \pm x_4 \pm \cdots \pm x_n$

ASSOCIATIVITY OF BINARY OPERATIONS

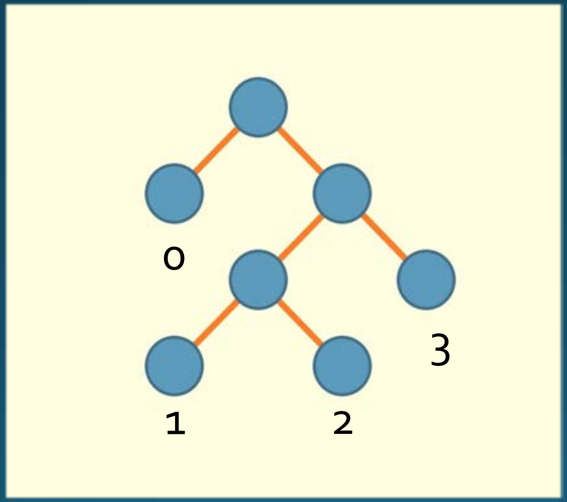
- Example:
  - $((x_0 - x_1) - x_2) - x_3 = x_0 - x_1 - x_2 - x_3$
  - $(x_0 - (x_1 - x_2)) - x_3 = x_0 - x_1 + x_2 - x_3$
  - $x_0 - ((x_1 - x_2) - x_3) = x_0 - x_1 + x_2 + x_3$
  - $x_0 - (x_1 - (x_2 - x_3)) = x_0 - x_1 + x_2 - x_3$
  - $(x_0 - x_1) - (x_2 - x_3) = x_0 - x_1 - x_2 + x_3$
- Question#1: $C_3=5$
- Question#2: $C_{-,3}=4$

DOUBLE MINUS OPERATIONS

- Definition: $\boldsymbol{A \ominus B = -A-B}$
- EXAMPLES :
  - $(A \ominus B) \ominus C = -(-A-B)-C = A+B-C$
  - $A \ominus (B \ominus C) = -A-(-B-C) = -A+B+C$
- <u>NOT Associative</u>
- Question#2?

BINARY TREES

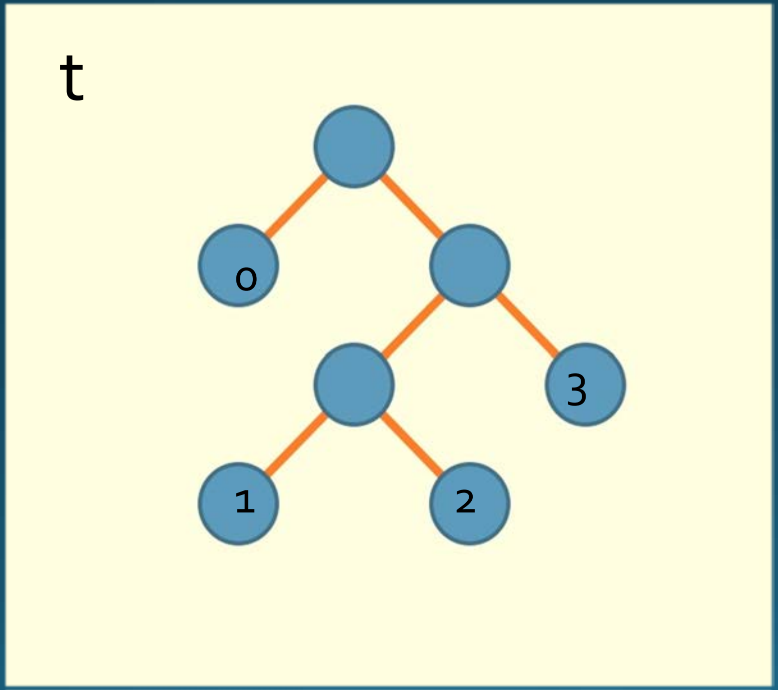
- **DEFINITION:** A full binary tree is defined as a tree in which every node other than the leaves has two children.



- **Leaf depth d(t)**: the numbers of steps needed from the top node down to the leaves

BINARY TREES

A ɵ ((B ɵ C) ɵ D)= *-A-(-(-B-C)-D)* = *-A-B-C+D*



Depths of Leaves d(t)=(1,3,3,2)

PARENTHESIZATIONS

- Define: $C_{\theta,n,r}$ is the number of distinct results from $X_0 \ominus X_1 \ominus X_2 \ominus X_3 \ominus \ldots\ldots \ominus X_{n-1} \ominus X_n$ with exactly r plus signs.
- For n=3, we have $X_0 \ominus X_1 \ominus X_2 \ominus X_3$
  - $C_{\theta,3,0}=0$
  - $C_{\theta,3,1}=4=\binom{4}{1}$ (+---,-+--,--+-,---+)
  - $C_{\theta,3,2}=0$
  - $C_{\theta,3,3}=0$
  - $C_{\theta,3,4}=1=\binom{4}{4}$ (++++)

PARENTHESIZATIONS

$C_{\ominus}, n, r$	0	1	2	3	4	5	6	7	8	9	10	11
$C_{\ominus}, 0, r$		$\binom{1}{1}$										
$C_{\ominus}, 1, r$	$\binom{2}{0}$											
$C_{\ominus}, 2, r$			$\binom{3}{2} - 1$									
$C_{\ominus}, 3, r$		$\binom{4}{1}$			$\binom{4}{4}$							
$C_{\ominus}, 4, r$	$\binom{5}{0}$			$\binom{5}{3} - 1$								
$C_{\ominus}, 5, r$			$\binom{6}{2}$			$\binom{6}{5}$						
$C_{\ominus}, 6, r$		$\binom{7}{1}$			$\binom{7}{4} - 1$			$\binom{7}{7}$				
$C_{\ominus}, 7, r$	$\binom{8}{0}$			$\binom{8}{3}$			$\binom{8}{6}$					
$C_{\ominus}, 8, r$			$\binom{9}{2}$			$\binom{9}{5} - 1$			$\binom{9}{8}$			
$C_{\ominus}, 9, r$		$\binom{10}{1}$			$\binom{10}{4}$			$\binom{10}{7}$			$\binom{10}{10}$	
$C_{\ominus}, 10, r$	$\binom{11}{0}$			$\binom{11}{3}$			$\binom{11}{6} - 1$			$\binom{11}{9}$		
$C_{\ominus}, 11, r$			$\binom{12}{2}$			$\binom{12}{5}$			$\binom{12}{8}$			$\binom{12}{11}$

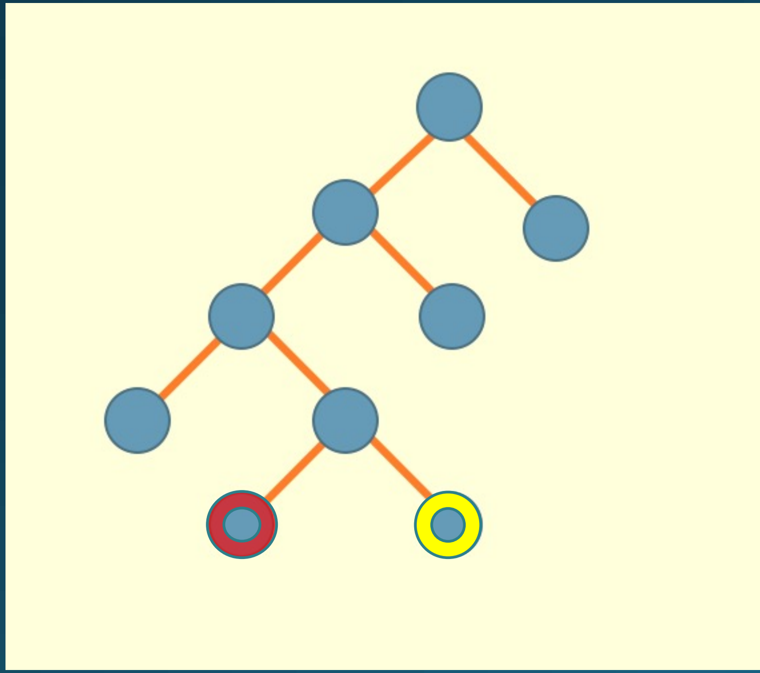
THEOREM (Huang, Mickey and X.)

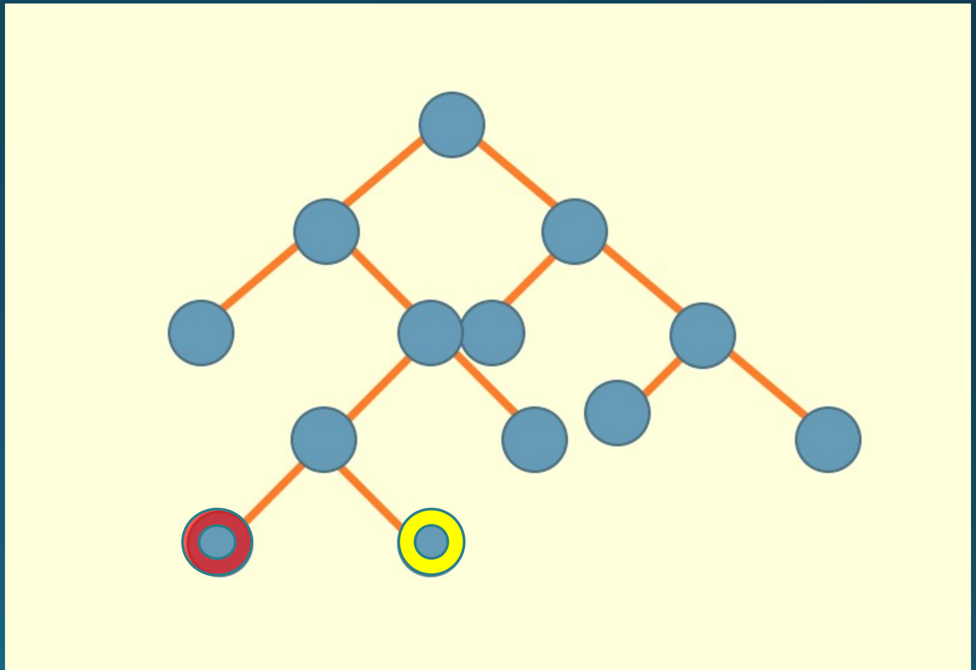
- For $n \geq 1$ *and* $0 \leq r \leq n+1$:

$$C_{\theta,n,r} = \begin{cases} \binom{n+1}{r} & if\ n+r \equiv 1 (mod\ 3)\ and\ n \neq 2r-2; \\ \binom{n+1}{r} - 1 & if\ n+r \equiv 1 (mod\ 3)\ and\ n = 2r-2; \\ 0 & if\ n+r \not\equiv 1 (mod\ 3). \end{cases}$$

WHY "-1"

- In every binary tree there exists a leaf with the same depth as the next leaf.



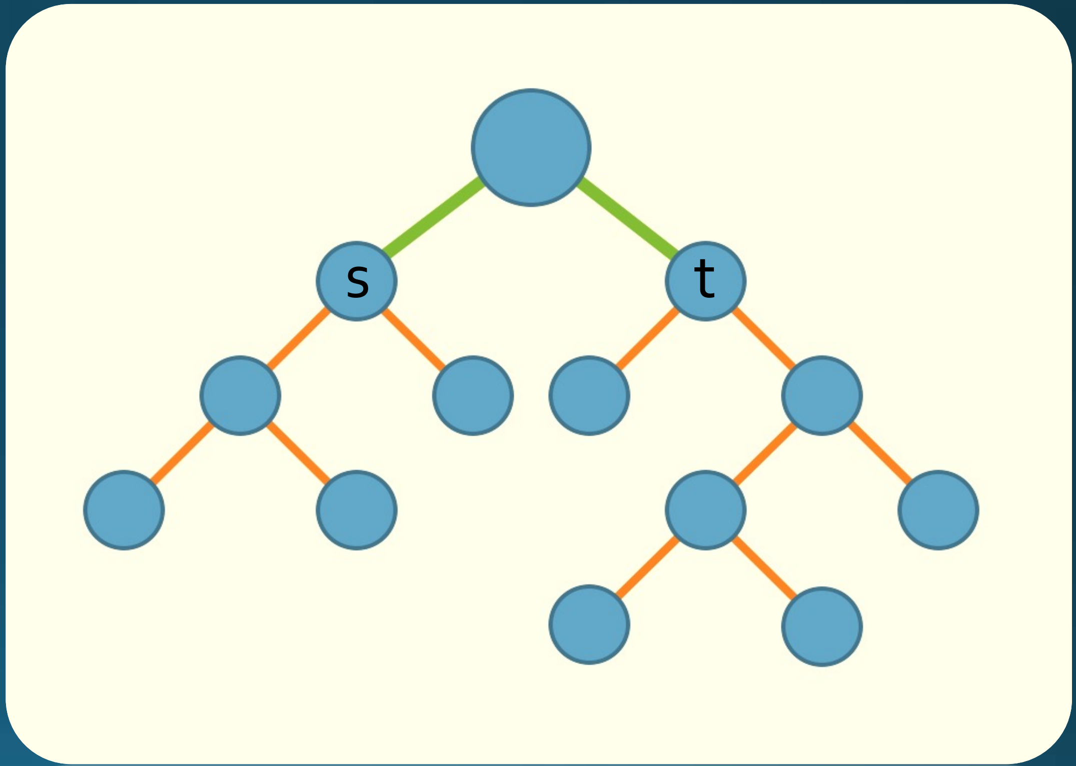


- So, +-+-+-… or -+-+-+… are impossible.

PROOF: INDUCTION ON TREE SIZE

- $d(s) = (2, 2, 1)$
- $d(t) = (1, 3, 3, 2)$

- $d(s \wedge t) = (2+1, 2+1, 1+1,$
- $1+1, 3+1, 3+1, 2+1)$
- $=(3, 3, 2, 2, 4, 4, 3)$



Thank you!