Hecke algebras with independent parameters

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Coxeter systems

Definition

- A **Coxeter group** is a group W generated by a finite set S with
  - quadratic relations $s^2 = 1$ for all $s \in S$, and
  - braid relations $(sts\cdots)_{m_{st}} = (tst\cdots)_{m_{st}}$ for all $s, t \in S$.
- The **Coxeter system** (W, S) is represented by its **Coxeter diagram**.

Example (Coxeter systems of type A and B)

- The **symmetric group** $\mathfrak{S}_n$ is generated by $s_1, \ldots, s_{n-1}$, where $s_i := (i, i+1)$ is a simple transposition. The Coxeter diagram is:

$$s_1 \text{ ——— } s_2 \text{ ——— } \cdots\cdots \text{ ——— } s_{n-2} \text{ ——— } s_{n-1}$$

- The **hyperoctahedral group** $\mathfrak{S}_n^{\pm}$ consists of **signed permutations** of $\{\pm 1, \ldots, \pm n\}$. It is generated by $s_1, \ldots, s_{n-1}$, and $s_0 = \overline{1}2\cdots n$.

$$s_0 \text{ ═══ } s_1 \text{ ——— } \cdots\cdots \text{ ——— } s_{n-2} \text{ ——— } s_{n-1}$$

(Iwahori-)Hecke algebras

Definition

- The **(Iwahori-)Hecke algebra** $\mathcal{H}_S(q)$ of a Coxeter system (W, S) is an algebra over a field $\mathbb{F}$ generated by $\{T_s : s \in S\}$ with
  - quadratic relations $(T_s - 1)(T_s + q) = 0$ for all $s \in S$, and
  - braid relations $(T_s T_t T_s \cdots)_{m_{st}} = (T_t T_s T_t \cdots)_{m_{st}}$ for all $s, t \in S$.
- The Hecke algebra of $\mathfrak{S}_n$ is denoted by $H_n(q)$.

Fact

- *Given $w \in W$ with any reduced expression $w = s_1 \cdots s_k$, define $T_w := T_{s_1} \cdots T_{s_k}$. Then $\{T_w : w \in W\}$ is a basis for $\mathcal{H}_S(q)$.*
- *The specialization of $\mathcal{H}_S(q)$ at $q = 1$ gives the group algebra $\mathbb{F}W$.*
- *When $\mathbb{F} = \mathbb{C}$ and $q \in \mathbb{C} \setminus \{0, \textit{roots of unity}\}$ then $H_S(q) \cong \mathbb{C}W$.*
- *The **0-Hecke algebra** $H_S(0)$ is a monoid algebra analogous to $\mathbb{F}W$.*
- *(Complex) representations of $\mathfrak{S}_n \leftrightarrow$ self-dual Hopf algebra Sym.*
- *Representations of $\mathcal{H}_n(0) \leftrightarrow$ dual Hopf algebras QSym and **NSym**.*

Hecke algebras with independent parameters

Definition

The Hecke algebra $\mathcal{H}_S(\mathbf{q}) = \mathcal{H}(\mathbf{q})$ of (W, S) with **independent parameters** $\mathbf{q} = (q_s \in \mathbb{F} : s \in S)$ is the $\mathbb{F}$-algebra generated by $\{T_s : s \in S\}$ with

- quadratic relations $(T_s - 1)(T_s + q_s) = 0$ for all $s \in S$, and
- braid relations $(T_s T_t T_s \cdots)_{m_{st}} = (T_t T_s T_t \cdots)_{m_{st}}$ for all $s, t \in S$.

Example

$$1 = 0 - 1 - 0 - 1 - 0 - 1 - 0$$

Fact

- *If $q_s = q$ for all $s \in S$ then $\mathcal{H}_S(\mathbf{q})$ is the Hecke algebra $H_S(q)$.*
- *If one only insists $q_s = q_t$ whenever m_{st} is odd, then $\mathcal{H}_S(\mathbf{q})$ is the* **Hecke algebra with unequal parameters** *in the sense of Lusztig.*

Theorem (Well-known to experts)

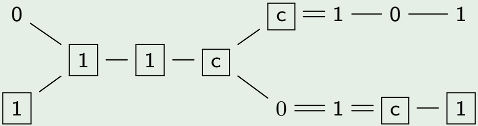
There is a spanning set $\{T_w : w \in W\}$ for $\mathcal{H}_S(\mathbf{q})$, which is indeed a basis if and only if $\mathcal{H}(\mathbf{q})$ is a Hecke algebra with unequal parameters.

Theorem (H.)

If there exist $s, t \in S$ with m_{st} odd such that q_s and q_t are distinct nonzero parameters, then one has $\mathcal{H}_S(\mathbf{q}) \cong \mathcal{H}_{S \setminus R}(\mathbf{q})$ where R consists of all elements $r \in S$ connected to s via some path with odd edge weights and nonzero vertex labels in the Coxeter diagram of (W, S).

Example (Collapse)

Let $\mathbb{F}$ be a field with at least 3 distinct elements 0, 1, and c. Let $\mathcal{H}_S(\mathbf{q})$ be given by the diagram below. Then R consists of the boxed elements.



Definition

The Hecke algebra $\mathcal{H}(\mathbf{q})$ is **collapse free** if one of q_s and q_t is 0 whenever m_{st} is odd and $q_s \neq q_t$.

Theorem (H.)

Assume (W, S) is simply laced and $\mathcal{H}(\mathbf{q})$ is collapse free. Then we can explicitly construct a basis for $\mathcal{H}(\mathbf{q})$ indexed by a subset of W.

Theorem (H.)

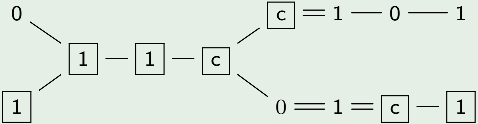
The algebra $\mathcal{H}(\mathbf{q})$ is collapse free and commutative if and only if (W, S) is simply laced and exactly one of q_s and q_t is 0 for all $s, t \in S$ with $m_{st} = 3$. The second condition implies the Coxeter diagram of (W, S) is bipartite.

Corollary (H.)

If $\mathcal{H}(\mathbf{q})$ is collapse free and commutative then its dimension is the **Merrifield-Simmons index** $|\mathcal{I}(G)|$ *of the underlying graph G of the Coxeter diagram of (W, S), where $\mathcal{I}(G)$ consists of independent sets in G.*

Example (Dimensions for commutative $\mathcal{H}(\mathbf{q})$)

- If (W, S) is of type A_n then $\dim_{\mathbb{F}} \mathcal{H}(\mathbf{q})$ is the Fibonacci number F_{n+2}.
- If (W, S) is of affine type $\widetilde{A}_n$ then $\dim_{\mathbb{F}} \mathcal{H}(\mathbf{q})$ is the **Lucas number** L_n defined by $L_0 = 2$, $L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$.
- We have seen a Hecke algebra $\mathcal{H}(\mathbf{q})$ given by the diagram below. Removing the boxed elements gives 3 connected components 0, $0 = 1$, and $1 - 0 - 1$. Thus $\dim_{\mathbb{F}} \mathcal{H}(\mathbf{q}) = 2 \cdot 8 \cdot 5 = 80$.



Conjecture (Proved for type A)

If the Coxeter diagram of (W, S) is a simply laced bipartite graph G, then a collapse-free $\mathcal{H}(\mathbf{q})$ has minimum dimension equal to $|\mathcal{I}(G)|$, which is attained when $\mathcal{H}(\mathbf{q})$ is commutative.

Generalization to all graphs

Definition

Commutative Hecke algebras with independent parameters can be generalized to certain algebras $\mathcal{H}(G, R)$, where G is any simple graph and R is any subset of vertices of G.

Proposition (H.)

- *$\mathcal{H}(G, R)$ is a commutative algebra with* $\dim_{\mathbb{F}} \mathcal{H}(G, R) = |\mathcal{I}(G)|$.
- *If $\mathcal{H}(\mathbf{q})$ is collapse free and commutative then $\mathcal{H}(\mathbf{q}) \cong \mathcal{H}(G, R)$ for G being the Coxeter diagram of (W, S) and $R = \{s \in S : q_s = -1\}$.*
- *The projective indecomposable $\mathcal{H}(G, R)$-modules are indexed by $\mathcal{I}(G - R)$.*
- *The simple $\mathcal{H}(G, R)$-modules are all one-dimensional and also indexed by $\mathcal{I}(G - R)$.*
- *The Cartan matrix of $\mathcal{H}(G, R)$ is a diagonal matrix.*
- *The algebra $\mathcal{H}(G, R)$ is semisimple if and only if $R = \emptyset$.*

Semisimple algebras in type A

Proposition (H.)

- *Let* $\mathcal{H}_n := \mathcal{H}(P_{n-1}, \emptyset)$*, where* $\mathcal{H}_0 := \mathbb{F}$ *by convention.*
- *If* $\mathrm{char}(\mathbb{F}) \neq 2$ *then* $\mathcal{H}_n \cong \mathcal{H}_n(0,1,0,1,\ldots) \cong \mathcal{H}_n(1,0,1,0,\ldots)$*.*
- *Write* $\alpha \propto n$ *if* α *is a composition of* n *with internal parts at least* 2*.*
- *Simple* $\mathcal{H}_n$*-modules* $\mathcal{C}_\alpha$ *are indexed by* $\alpha \propto n$*, all one-dimensional.*
- *The* **Grothendieck group** *of the tower* $\mathcal{H}_\bullet : \mathcal{H}_0 \hookrightarrow \mathcal{H}_1 \hookrightarrow \mathcal{H}_2 \hookrightarrow \cdots$ *is the free abelian group* $G_0(\mathcal{H}_\bullet)$ *with a basis* $\sqcup_{n\geq 0}\{\mathcal{C}_\alpha : \alpha \propto n\}$*.*
- *There is a product and a coproduct of* $G_0(\mathcal{H}_\bullet)$ *defined by*

$$\mathcal{C}_\alpha \,\widehat{\otimes}\, \mathcal{C}_\beta := (\mathcal{C}_\alpha \otimes \mathcal{C}_\beta)\uparrow_{\mathcal{H}_m\otimes\mathcal{H}_n}^{\mathcal{H}_{m+n}} \quad \text{and} \quad \Delta(\mathcal{C}_\alpha) := \sum_{0\leq i\leq m} \mathcal{C}_\alpha \downarrow_{\mathcal{H}_i\otimes\mathcal{H}_{m-i}}^{\mathcal{H}_m}.$$

Proposition (H.)

$G_0(\mathcal{H}_\bullet)$ has a self-dual basis $\{\mathcal{C}_\alpha : \alpha \propto n,\ \forall n \geq 0\}$. For $\alpha \propto m$ and $\beta \propto n$,

$$\mathcal{C}_\alpha \,\widehat{\otimes}\, \mathcal{C}_\beta = \begin{cases} \mathcal{C}_{\alpha\beta} \oplus \mathcal{C}_{\alpha \rhd \beta}, & \text{if } \alpha\beta \propto m+n, \\ \mathcal{C}_{\alpha \rhd \beta}, & \text{otherwise}, \end{cases} \quad \textit{and}$$

$$\Delta(\mathcal{C}_\alpha) = \sum_{0 \leq i \leq m} \mathcal{C}_{\alpha_{\leq i}} \otimes \mathcal{C}_{\alpha_{>i}}.$$

Example

One has $\mathcal{C}_{132} \,\widehat{\otimes}\, \mathcal{C}_{41} = \mathcal{C}_{13241} \oplus \mathcal{C}_{1361}$, $\mathcal{C}_{121} \,\widehat{\otimes}\, \mathcal{C}_{32} = \mathcal{C}_{1242}$, and

$$\Delta(\mathcal{C}_{122}) = \mathcal{C}_\emptyset \otimes \mathcal{C}_{122} + \mathcal{C}_1 \otimes \mathcal{C}_{22} + \mathcal{C}_{11} \otimes \mathcal{C}_{12} + \mathcal{C}_{12} \otimes \mathcal{C}_2 + \mathcal{C}_{121} \otimes \mathcal{C}_1 + \mathcal{C}_{122} \otimes \mathcal{C}_\emptyset.$$

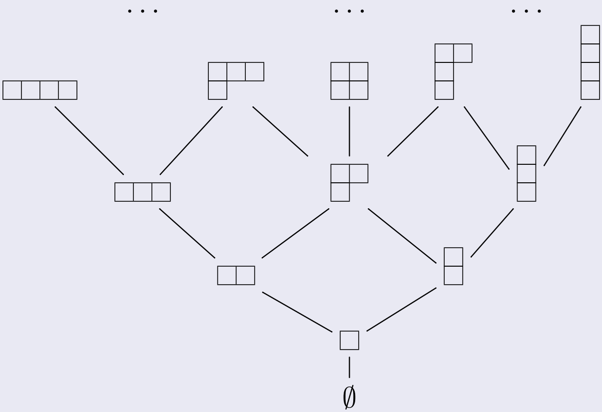
Corollary (H.)

There is a surjection $K_0(\mathcal{H}_\bullet(0)) \twoheadrightarrow G_0(\mathcal{H}_\bullet)$ of graded algebras and a dual injection $G_0(\mathcal{H}_\bullet) \hookrightarrow G_0(\mathcal{H}_\bullet(0))$ of graded coalgebras.

Bratteli diagram

Fact (Young's lattice)

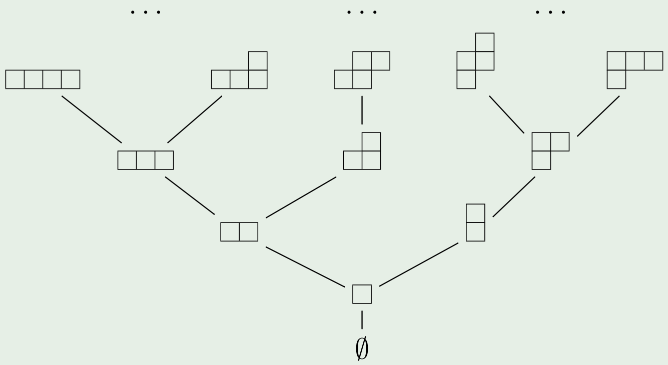
The Bratteli diagram of the tower $\mathfrak{S}_0 \subseteq \mathfrak{S}_1 \subseteq \mathfrak{S}_2 \subseteq \cdots$ of symmetric groups is the Young's lattice.



Corollary (H.)

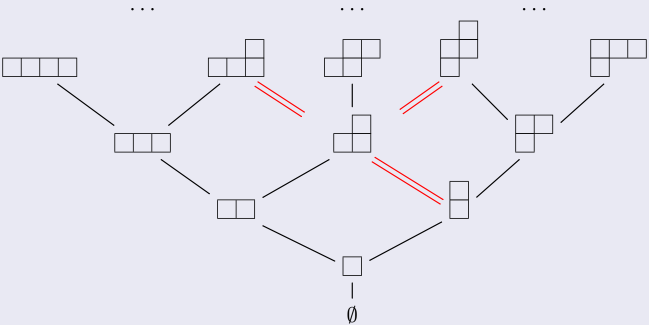
The Bratteli diagram of the tower $\mathcal{H}_\bullet : \mathcal{H}_0 \hookrightarrow \mathcal{H}_1 \hookrightarrow \mathcal{H}_2 \hookrightarrow \cdots$ *is a binary tree consisting of compositions with internal parts larger than* 1*.*

Example (Bratteli Diagram of $\mathcal{H}_\bullet$)



Remark (Young-Fibonacci lattice)

The Bratteli diagram of $\mathcal{H}_\bullet$ is a lattice weaker than the **Young-Fibonacci lattice**.



Remark

Okada defined a different tower of algebras of dimensions $n!$ whose Bratteli diagram is the Young-Fibonacci lattice.

Antipodes

Fact

- *The antipode of* Sym *is given by* $s_\lambda \mapsto (-1)^{|\lambda|} s_{\lambda^t}$.
- *The antipode of* QSym *is given by* $F_\alpha \mapsto (-1)^{|\alpha|} F_{\alpha^t}$.
- *The antipode of* **NSym** *is given by* $\mathbf{s}_\alpha \mapsto (-1)^{|\alpha|} \mathbf{s}_{\alpha^t}$.

Proposition (H.)

The antipode of $G_0(\mathcal{H}_\bullet)$ *is given by* $\mathcal{C}_\alpha \mapsto (-1)^{|\alpha|} \mathcal{C}_{\alpha^c}$, $\forall \alpha \propto n$, $\forall n \geq 0$.

Example

- The composition $\alpha = 21321$ has descent set $D(\alpha) = \{2, 3, 6, 8\} \subseteq [8]$.
- Its complement is $\alpha^c = 13122$ with descent set $D(\alpha)^c = \{1, 4, 5, 7\}$.
- The conjugate/transpose of α is $\alpha^t = 22131$ (the reverse of α^c).

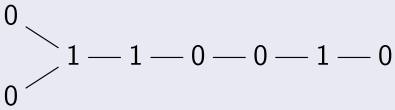
Questions

Question

- *What is the representation theory of a Hecke algebras of type A with independent parameters?*

$$0 - 0 - 1 - 1 - 1 - 0 - 1 - 1$$

- *What is the representation theory of a Hecke algebra of a simply laced Coxeter system* (W, S) *with independent parameters?*



- *What if* (W, S) *is not simply laced?*

Thank you!