0-Hecke algebra actions on quotients of polynomial rings

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The Symmetric Group $\mathfrak{S}_n$

- The *symmetric group* $\mathfrak{S}_n := \{\text{bijections on } \{1,\ldots,n\}\}$ is generated by the *adjacent transpositions* $s_i = (i, i+1)$, $1 \le i \le n-1$, with quadratic relations $s_i^2 = 1$, $1 \le i \le n-1$, and braid relations

$$\begin{cases} s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, & 1 \le i \le n-2, \\ s_i s_j = s_j s_i, & |i-j| > 1. \end{cases}$$

- More generally, a *Coxeter group* has a similar presentation.
- The *length* of any $w \in \mathfrak{S}_n$ is $\ell(w) := \min\{k : w = s_{i_1} \cdots s_{i_k}\}$, which coincides with $\mathrm{inv}(w) := \{(i,j) : 1 \le i < j \le n, w(i) > w(j)\}$.
- For example, $w = 3241 \in \mathfrak{S}_4$ has $\ell(w) = \mathrm{inv}(w) = 4$ and reduced repressions $w = s_2 s_1 s_2 s_3 = s_1 s_2 s_1 s_3 = s_1 s_2 s_3 s_1$.

The Hecke Algebra $H_n(q)$

- The *(Iwahori-)Hecke algebra* $H_n(q)$ is a deformation of the group algebra $\mathbb{F}\mathfrak{S}_n$ of $\mathfrak{S}_n$ over an arbitrary field $\mathbb{F}$.
- It is an $\mathbb{F}(q)$-algebra generated by $T_1, \ldots, T_{n-1}$ with relations

$$\begin{cases} (T_i+1)(T_i-q)=0, & 1 \le i \le n-1, \\ T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, & 1 \le i \le n-2, \\ T_i T_j = T_j T_i, & |i-j|>1. \end{cases}$$

- It has an $\mathbb{F}(q)$-basis $\{T_w : w \in \mathfrak{S}_n\}$, where $T_w := T_{s_1} \cdots T_{s_k}$ if $w = s_1 \cdots s_k$ with k minimum.
- It has significance in algebraic combinatorics, knot theory, quantum groups, representation theory of p-adic groups, etc.

The 0-Hecke algebra $H_n(0)$

- Set $q=1$: $H_n(q) \to \mathbb{F}\mathfrak{S}_n$, $T_i \to s_i$, $T_w \to w$.
- Tits showed that $H_n(q) \cong \mathbb{C}\mathfrak{S}_n$ unless $q \in \{0, \text{roots of unity}\}$.
- Set $q=0$: $H_n(q) \to H_n(0)$, $T_i \to \overline{\pi}_i$, $T_w \to \overline{\pi}_w$,

$$\begin{cases} \overline{\pi}_i^2 = -\overline{\pi}_i, & 1 \le i \le n-1, \\ \overline{\pi}_i\overline{\pi}_{i+1}\overline{\pi}_i = \overline{\pi}_{i+1}\overline{\pi}_i\overline{\pi}_{i+1}, & 1 \le i \le n-2, \\ \overline{\pi}_i\overline{\pi}_j = \overline{\pi}_j\overline{\pi}_i, & |i-j|>1. \end{cases}$$

- $H_n(0)$ has another generating set $\{\pi_i := \overline{\pi}_i + 1\}$, with relations

$$\begin{cases} \pi_i^2 = \pi_i, & 1 \le i \le n-1, \\ \pi_i\pi_{i+1}\pi_i = \pi_{i+1}\pi_i\pi_{i+1}, & 1 \le i \le n-2, \\ \pi_i\pi_j = \pi_j\pi_i, & |i-j|>1. \end{cases}$$

- Sending π_i to $-\overline{\pi}_i$ gives an algebra automorphism.

Significance of the 0-Hecke algebra

- Using the automorphism $\pi_i \mapsto -\overline{\pi}_i$ of $H_n(0)$, Stembridge (2007) gave a short derivation for the Möbius function of the *Bruhat order* of $\mathfrak{S}_n$ (or more generally, any Coxeter group).
- Norton (1979) studied the representation theory of $H_n(0)$ over an arbitrary field $\mathbb{F}$.
- Norton's result provides motivations to work of Denton, Hivert, Schilling, and Thiéry (2011) on the representation theory of finite *$\mathcal{J}$-trivial monoids*.
- Krob and Thibon (1997) discovered connections between $H_n(0)$-representations and certain generalizations of symmetric functions, which is similar to the classical Frobenius correspondence between $\mathfrak{S}_n$-representations and symmetric functions.

Analogies between $\mathfrak{S}_n$ and $H_n(0)$

- $\mathbb{F}\mathfrak{S}_n$ is the group algebra of the symmetric group $\mathfrak{S}_n$ and $H_n(0)$ is the monoid algebra of the monoid $\{\pi_w : w \in W\}$.
- The defining representations of $\mathfrak{S}_n$ and $H_n(0)$ are analogous:

$$1 \xleftrightarrow{s_1} 2 \xleftrightarrow{s_2} \cdots \xleftrightarrow{s_{n-1}} n$$

$$1 \xrightarrow{\pi_1} 2 \xrightarrow{\pi_2} \cdots \xrightarrow{\pi_{n-1}} n$$

- $\mathfrak{S}_n$ acts on $\mathbb{Z}^n$: s_i swaps a_i and a_{i+1} in $a_1 \cdots a_n$.
- $H_n(0)$ acts on $\mathbb{Z}^n$ by the *bubble-sorting operators*: π_i swaps a_i and a_{i+1} in $a_1 \cdots a_n$ if $a_i > a_{i+1}$, or fixes $a_1 \cdots a_n$ otherwise.
- Analogies between other representations of $\mathfrak{S}_n$ and $H_n(0)$?

Actions on polynomials

- $\mathfrak{S}_n$ acts on $\mathbb{F}[X] := \mathbb{F}[x_1, \ldots, x_n]$ by variable permutation.
- $H_n(0)$ also acts on $\mathbb{F}[X]$ via the *Demazure operators*

$$\pi_i(f) := \partial_i(x_i f) = \frac{x_i f - s_i(x_i f)}{x_i - x_{i+1}}.$$

- The *divided difference operator* ∂_i is useful in Schubert calculus, a branch of algebraic geometry.
- $\pi_1(x_1^3 x_2 x_3 x_4^4) = (x_1^3 x_2 + x_1^2 x_2^2 + x_1 x_2^3) x_3 x_4^4$.
- $\pi_2(x_1^3 x_2 x_3 x_4^4) = x_1^3 x_2 x_3 x_4^4$.
- $\pi_3(x_1^3 x_2 x_3 x_4^4) = x_1^3 x_2 (-x_3^2 x_4^3 - x_3^3 x_4^2)$.

The coinvariant algebra of $\mathfrak{S}_n$

- The *invariant ring* $\mathbb{F}[X]^{\mathfrak{S}_n} := \{f \in \mathbb{F}[X] : wf = f, \forall w \in \mathfrak{S}_n\}$ consists of all symmetric functions in $x_1, \ldots, x_n$. It is a polynomial ring $\mathbb{F}[X]^{\mathfrak{S}_n} = \mathbb{F}[e_1, \ldots, e_n]$ in the *elementary symmetric functions*

$$e_k := \sum_{1 \le i_1 < \cdots < i_k \le n} x_{i_1} \cdots x_{i_k}, \quad k = 1, \ldots, n.$$

$n = 3$: $e_1 = x_1 + x_2 + x_3$, $e_2 = x_1x_2 + x_1x_3 + x_2x_3$, $e_3 = x_1x_2x_3$

- If $f \in \mathbb{F}[X]^{\mathfrak{S}_n}$ and $g \in \mathbb{F}[X]$, then $s_i(fg) = fs_i(g)$.
- Thus $\mathbb{F}[X]/(e_1, \ldots, e_n)$ becomes a graded $\mathfrak{S}_n$-module.

Theorem (Chevalley–Shephard–Tod 1955, indirect proof)

The coinvariant algebra $\mathbb{F}[X]/(e_1, \ldots, e_n)$ is isomorphic to the regular representation $\mathbb{F}\mathfrak{S}_n$ of $\mathfrak{S}_n$, if $\mathbb{F}$ is a field of characteristic 0.

The coinvariant algebra of $H_n(0)$

- The $H_n(0)$-invariants are also the symmetric functions: $\pi_i f = f$ if and only if $s_i f = f$ for all i.
- If $f \in \mathbb{F}[X]^{\mathfrak{S}_n}$ and $g \in \mathbb{F}[X]$, then $\pi_i(fg) = f\pi_i(g)$.
- Thus $\mathbb{F}[X]/(e_1, \ldots, e_n)$ becomes a graded $H_n(0)$-module.

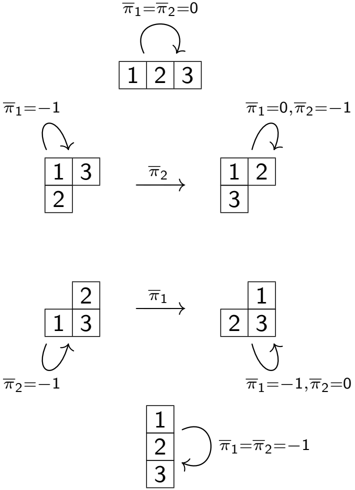
Theorem (H. 2014)

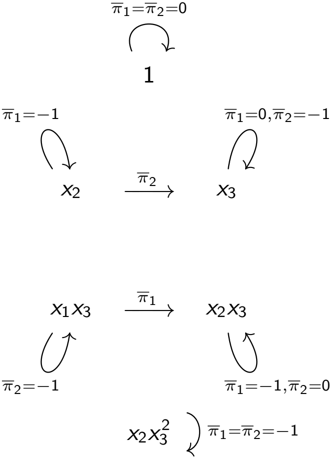
The coinvariant algebra $\mathbb{F}[X]/(e_1, \ldots, e_n)$ is isomorphic to the regular representation of $H_n(0)$.

Remark

Our proof is constructive, using the *descent basis* of the coinvariant algebra given by Garsia and Stanton (1984).

$H_3(0) \cong \mathbb{F}[x_1, x_2, x_3]/(e_1, e_2, e_3)$





$$\mathbf{ch}_t = \mathbf{s}_3 + t\mathbf{s}_{12} + t^2\mathbf{s}_{21} + t^3\mathbf{s}_{111}$$

Representation theory of $\mathfrak{S}_n$

- Every $\mathfrak{S}_n$-module is a direct sum of simple modules.
- A *partition* of n is a decreasing sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$ of positive integers whose sum is n; this is denoted by $\lambda \vdash n$.
- The simple $\mathfrak{S}_n$-modules S^λ are indexed by partitions $\lambda \vdash n$.
- The *Schur function* s_λ is the sum of x_τ for all *semistandard tableaux* τ of shape λ. For example,

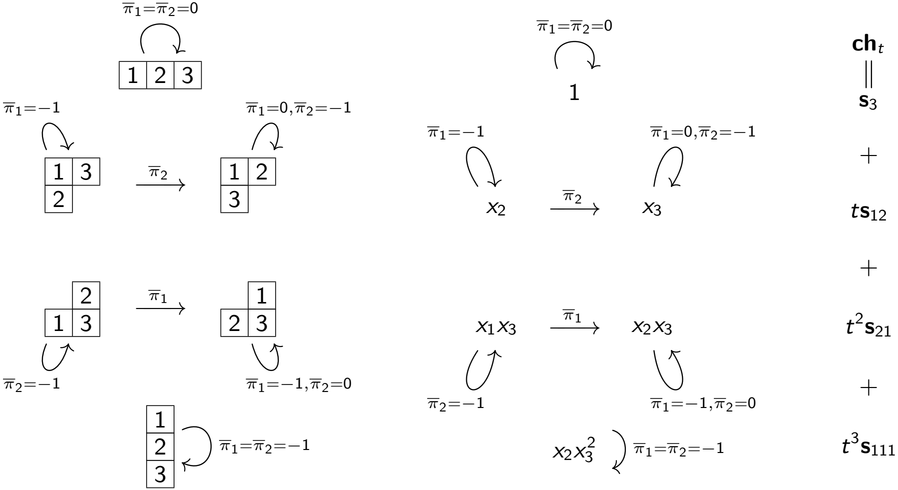
$$s_{21} = x_{\begin{smallmatrix}1&1\\2&\end{smallmatrix}} + x_{\begin{smallmatrix}1&2\\2&\end{smallmatrix}} + \cdots = x_1^2 x_2 + x_1 x_2^2 + \cdots .$$

- Symmetric functions form a graded Hopf algebra with a self-dual basis $\{s_\lambda\}$.
- The *Frobenius characteristic map* $S^\lambda \mapsto s_\lambda$ is an isomorphism from $\mathfrak{S}_n$-representations to Sym.

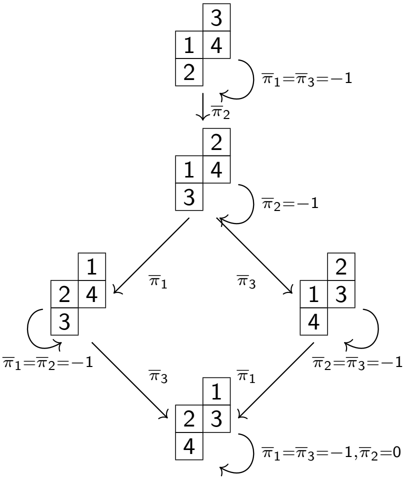
Representation theory of $H_n(0)$

- A *composition* of n, denoted by $\alpha \models n$, is a sequence $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ of positive integers whose sum is n.
- Norton (1979) showed that $H_n(0) = \bigoplus_{\alpha \models n} \mathbf{P}_\alpha$, so every projective indecomposable $H_n(0)$-module is isomorphic to $\mathbf{P}_\alpha$ for some $\alpha \models n$.
- Furthermore, every simple $H_n(0)$-module is isomorphic to some $\mathbf{C}_\alpha := \mathrm{top}(\mathbf{P}_\alpha) = \mathbf{P}_\alpha / \mathrm{rad}\, \mathbf{P}_\alpha$, which is 1-dimensional.
- Generalizing Sym are two graded Hopf algebras QSym (*quasisymmetric functions*) and **NSym** (*noncommutative symmetric functions*) with dual bases $\{F_\alpha\}$ and $\{\mathbf{s}_\alpha\}$. We have $\mathbf{NSym} \twoheadrightarrow \mathrm{Sym} \hookrightarrow \mathrm{QSym}$.
- Krob and Thibon (1997): by $\mathbf{P}_\alpha \mapsto \mathbf{s}_\alpha$ and $\mathbf{C}_\alpha \mapsto F_\alpha$ one has
  - $\{H_n(0)\text{-modules}\} \leftrightarrow \mathrm{QSym}$ (up to composition factors),
  - $\{\text{projective } H_n(0)\text{-modules}\} \leftrightarrow \mathbf{NSym}$.

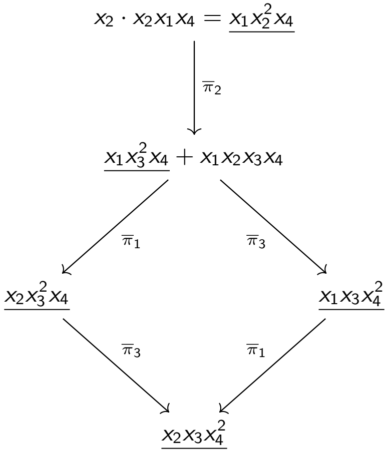
$H_3(0) \cong \mathbb{F}[x_1, x_2, x_3]/(e_1, e_2, e_3)$



$\alpha = (1, 2, 1)$



$\overline{\pi}_1 = \overline{\pi}_3 = -1$
$\overline{\pi}_2$
$\overline{\pi}_2 = -1$
$\overline{\pi}_1$
$\overline{\pi}_3$
$\overline{\pi}_1 = \overline{\pi}_2 = -1$
$\overline{\pi}_2 = \overline{\pi}_3 = -1$
$\overline{\pi}_3$
$\overline{\pi}_1$
$\overline{\pi}_1 = \overline{\pi}_3 = -1, \overline{\pi}_2 = 0$



$x_2 \cdot x_2 x_1 x_4 = \underline{x_1 x_2^2 x_4}$
$\overline{\pi}_2$
$\underline{x_1 x_3^2 x_4} + x_1 x_2 x_3 x_4$
$\overline{\pi}_1$
$\overline{\pi}_3$
$\underline{x_2 x_3^2 x_4}$
$\underline{x_1 x_3 x_4^2}$
$\overline{\pi}_3$
$\overline{\pi}_1$
$\underline{x_2 x_3 x_4^2}$

A generalization of the coinvariant algebra

- Let $n \geq k \geq 1$ be two integers. Define a homogeneous ideal

$$I_{n,k} := \langle x_1^k, x_2^k, \ldots, x_n^k, e_n, e_{n-1}, \ldots, e_{n-k+1} \rangle.$$

- The span of $x_1^k, x_2^k, \ldots, x_n^k$ is isomorphic to the defining representation of $\mathfrak{S}_n$.

$$1 \xleftrightarrow{s_1} 2 \xleftrightarrow{s_2} \cdots \xleftrightarrow{s_{n-1}} n$$

$$x_1^k \xleftrightarrow{s_1} x_2^k \xleftrightarrow{s_2} \cdots \xleftrightarrow{s_{n-1}} x_n^k$$

- The quotient $R_{n,k} := \mathbb{C}[X]/I_{n,k}$ is a graded $\mathfrak{S}_n$-module.
- The coinvariant algebra $\mathbb{C}[X]/(e_1, \ldots, e_n)$ is $R_{n,n}$.

The $\mathfrak{S}_n$-module structure of $R_{n,k}$

- Let $\mathcal{OP}_{n,k}$ be the set of all k-block partitions of the set $[n]$. For example, $(35|126|4) \in \mathcal{OP}_{6,3}$.
- We have $|\mathcal{OP}_{n,k}| = k! \cdot \mathrm{Stir}(n,k)$, where $\mathrm{Stir}(n,k)$ is the *(signless) Stirling number of the second kind*.
- Let $\mathrm{SYT}(n)$ be the set of *standard Young tableaux* of size n.

Theorem (Haglund–Rhoades–Shimozono 2017+)

As an ungraded $\mathfrak{S}_n$-module, $R_{n,k}$ is isomorphic to $\mathbb{C}[\mathcal{OP}_{n,k}]$. Moreover, the graded Frobenius characteristic of $R_{n,k}$ is

$$\sum_{\tau \in \mathrm{SYT}(n)} q^{\mathrm{maj}(\tau)} \binom{d - \mathrm{des}(\tau) - 1}{n-k}_q s_{\mathrm{shape}(\tau)}.$$

A 0-Hecke analogue

- Define $J_{n,k}$ to be the ideal of $\mathbb{F}[X]$ generated by elementary symmetric functions $e_n, e_{n-1}, \ldots, e_{n-k+1}$ and *complete homogeneous symmetric functions* $h_k(x_1), h_k(x_1, x_2), \ldots, h_k(x_1, x_2, \ldots, x_n)$.
- The span of $h_k(x_1), h_k(x_1, x_2), \ldots, h_k(x_1, x_2, \ldots, x_n)$ is isomorphic to the defining representation of $H_n(0)$.

$$1 \xrightarrow{\pi_1} 2 \xrightarrow{\pi_2} \cdots \xrightarrow{\pi_{n-1}} n$$

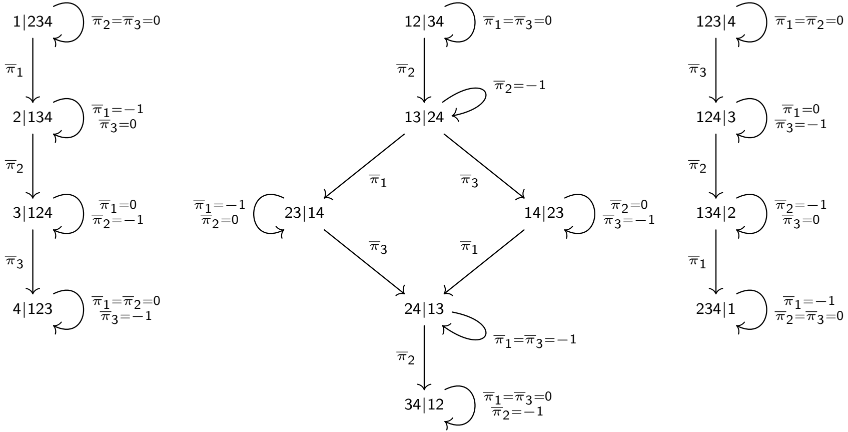
$$h_k(x_1) \xrightarrow{\pi_1} h_k(x_1, x_2) \xrightarrow{\pi_2} \cdots \xrightarrow{\pi_{n-1}} h_k(x_1, \ldots, x_n)$$

- The quotient $S_{n,k} := \mathbb{F}[X]/J_{n,k}$ is a graded $H_n(0)$-module.

Theorem (H.–Rhoades 2017+)

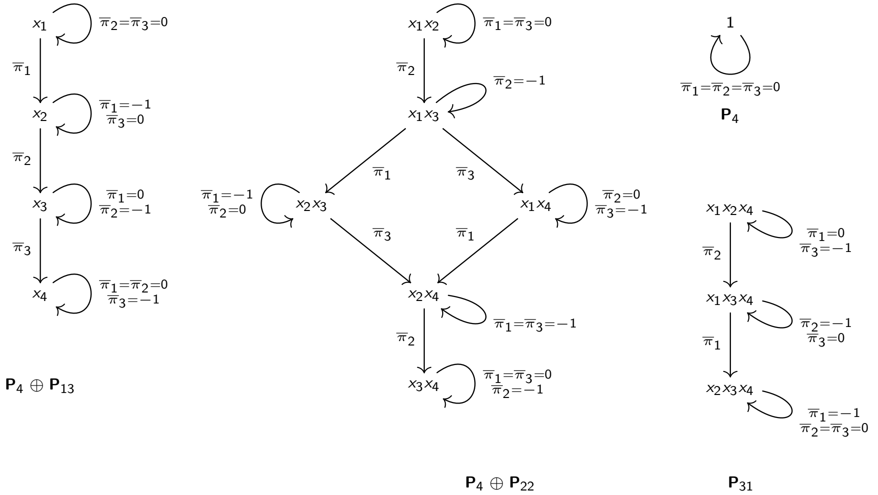
As an ungraded $H_n(0)$-module, $S_{n,k}$ is isomorphic to $\mathbb{F}[\mathcal{OP}_{n,k}]$.

A decomposition of $\mathbb{F}[\mathcal{OP}_{4,2}]$



$\mathcal{OP}_{13} \cong \mathbf{P}_4 \oplus \mathbf{P}_{13}$ $\mathcal{OP}_{22} \cong \mathbf{P}_4 \oplus \mathbf{P}_{22}$ $\mathcal{OP}_{31} \cong \mathbf{P}_4 \oplus \mathbf{P}_{31}$

A decomposition of $S_{4,2}$



Graded characteristics of $S_{n,k}$

Theorem (H.–Rhoades 2017+)

The graded $H_n(0)$-module $S_{n,k}$ corresponds

$$\sum_{\alpha \models n} t^{\mathrm{maj}(\alpha)} \begin{bmatrix} n-\ell(\alpha) \\ k-\ell(\alpha) \end{bmatrix}_t \mathbf{s}_\alpha \quad \textit{inside } \mathbf{NSym}$$

and its graded quasisymmetric characteristic coincides with the graded Frobenius characteristics of the $\mathfrak{S}_n$-module $R_{n,k}$.

Remark

This result connects to the *Delta Conjecture* of Haglund, Remmel, and Wilson (2016) in the theory of Macdonald polynomials.

More quotients of the polynomial ring

Theorem (DeConcini, Garsia, Procesi, Hotta, Springer, Tanisaki)

- *For any $\mu \vdash n$, $\mathbb{C}[X]$ has a homogeneous $\mathfrak{S}_n$-stable ideal J_μ generated by certain elementary symmetric functions in partial variable sets.*
- *$R_\mu = \mathbb{C}[X]/J_\mu$ is isomorphic to the cohomology ring of the Springer fiber indexed by μ.*
- *The graded Frobenius characteristic of $R_\mu = \mathbb{C}[X]/J_\mu$ is the modified Hall-Littlewood symmetric function*

$$\widetilde{H}_\mu(x;t) = \sum_\lambda t^{n(\mu)} K_{\lambda\mu}(t^{-1}) s_\lambda$$

where $n(\mu) = \mu_2 + 2\mu_3 + 3\mu_4 + \cdots$ and $K_{\lambda\mu}(t)$ is the Kostka-Foulkes polynomial.

$H_n(0)$-action on $R_\mu = \mathbb{C}[X]/J_\mu$

Theorem (H. 2014)

- *The ideal J_μ is $H_n(0)$-stable if and only if $\mu = (1^k, n-k)$ is a hook. Assume μ is a hook below.*
- *Then $R_\mu = \mathbb{C}[X]/J_\mu$ becomes a projective $H_n(0)$-module.*
- *Its graded noncommutative characteristic is*

$$\mathbf{ch}_t(\mathbb{C}[X]/J_\mu) = \sum_{\alpha \text{ refined by } \mu} t^{\mathrm{maj}(\alpha)} \mathbf{s}_\alpha = \widetilde{\mathbf{H}}_\mu(x;t).$$

- *Its graded quasisymmetric characteristic is*

$$\mathrm{Ch}_t(\mathbb{C}[X]/J_\mu) = \sum_{\alpha \text{ refined by } \mu} t^{\mathrm{maj}(\alpha)} s_\alpha = \widetilde{H}_\mu(x;t).$$

Stanley-Reisner ring of the Boolean algebra

- We introduced $H_n(0)$-actions on certain quotients of the Stanley-Reisner ring of the Boolean algebra [H. 2015].
- This gives multigraded $H_n(0)$-modules which correspond to
  - noncommutative analogues of $\widetilde{H}_\mu(x;t)$ introduced by Bergeron–Zabrocki (2005) and Lascoux–Novelli–Thibon (2013),
  - quasisymmetric generating function of the joint distribution of five permutation statistics studied by Garsia and Gessel (1979).
- We studied the Stanley-Reisner ring of the Coxeter complex of any finite Coxeter group.
- We are currently investigate a two-parameter family of quotients of the Stanley-Reisner ring (with Brendon Rhoades and Daniël Kroes).
- Is there a nice $H_n(0)$-action on the Stanley-Reisner ring of the Tits building of a finite general linear group?

Thank you!