Critical groups for Hopf algebra modules

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This is joint work with Darij Grinberg (UMN) and Victor Reiner (UMN).

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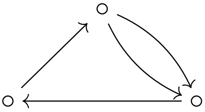

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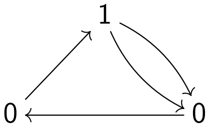

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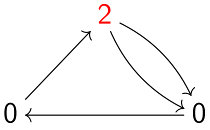

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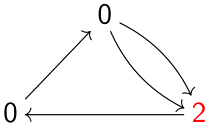

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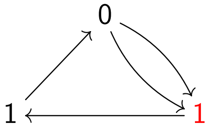

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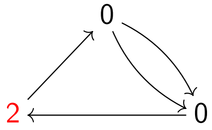

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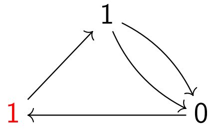

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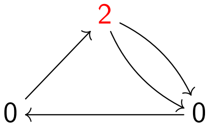

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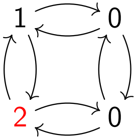

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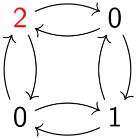

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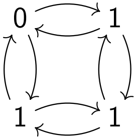

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- The order of the sandpile group is the number of directed spanning trees in which the sink is reachable from every vertex by a path.

A matrix point of view

- The Laplacian L of a digraph with vertices $1, 2, \ldots, n$ is an n-by-n matrix whose (i, j)-entry is

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- How to test whether a Z-matrix is avalanche-finite?

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Theorem (Gabrielov, Plemmons, Benkart–Klivans–Reiner)

Given a Z-matrix $C \in \mathbb{Z}^{\ell \times \ell}$, the following statements are equivalent.

1. *C is avalanche-finite.*
2. *C^t is avalanche-finite.*
3. *C is a nonsingular M-matrix.*
4. *There exists a column vector $x \in \mathbb{R}^\ell$ with $x > 0$ and $Cx > 0$.*
5. *Every eigenvalue of C has a positive real part.*

There are dozens of other statements equivalent to the above ones.

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- How can we find interesting avalanche-finite matrices?

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- Let $M_V := (m_{ij})_{i,j=0}^{\ell}$ and $L_V := \dim(V) \cdot I_{\ell+1} - M_V$.

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- Fix $V = D^{31}$. Then

$$M_V = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad L_V = \begin{pmatrix} 3 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & -1 & 0 \\ 0 & -1 & 3 & -1 & 0 \\ 0 & -1 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 3 \end{pmatrix}.$$

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- For $V = D^{31}$ we have $K(V) = \mathbb{Z}/4\mathbb{Z}$ since

$$\overline{L}_V = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & 0 \\ -1 & -1 & 2 & -1 \\ 0 & 0 & -1 & 3 \end{pmatrix} \text{ has Smith normal form } \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

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- In particular, the vector $\mathbf{s} := (\dim(S_0), \ldots, \dim(S_\ell))^t$ lies in $\ker(L_V)$.

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- How about other reflection groups?

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- How to define tensor product and trivial representation?

Group representations revisited

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- The group algebra $A = \mathbb{F}G$ becomes a Hopf algebra with the above coalgebra sturcture and an extra antipode $\alpha : \mathbb{F}G \to \mathbb{F}G$, $g \mapsto g^{-1}$.

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- An A-module V has two dual A-modules ${}^*V$ and V^* defined by

$$a(f)(v) := \begin{cases} f(\alpha(a)v), & f \in \mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F}) = V^*, \\ f(\alpha^{-1}(a)(v), & f \in \mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F}) = {}^*V. \end{cases}$$

Restricted universal enveloping Lie algebra

- The *Universal enveloping algebra* $\mathfrak{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is

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- Define the *critical group* of V to be $K(V) := \mathbf{s}^\perp / \operatorname{im}(L_V)$.

When is $K(V)$ finite?

Theorem (Grinberg, H. and Reiner)

The following are equivalent.

- $\overline{L_V}$ *is a nonsingular M-matrix.*
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Question

- How to test tensor-richness using some kind of character theory of A?
- Can we describe $\mathrm{rank}(L_V)$ using simple A-modules in $V^{\otimes k}$ for $k \geq 1$?

More results on $K(V)$

Let $d := \dim(A)$, $\mathbf{p} := (\dim(P_0), \ldots, \dim(P_\ell))$, and $\gamma := \gcd(\mathbf{p})$.

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- *Assume* $K(V)$ *is finite so that* L_V *has nonzero eigenvalues* $\lambda_1, \ldots, \lambda_\ell$. *Then* $|K(V)| = |\lambda_1 \cdots \lambda_\ell \cdot \gamma / d|$.

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- *We have* $\mathbf{s} \in \ker((L_V)^t)$ *and* $\mathbf{p} \in \ker(L_V)$. *(To prove the latter we need both* ${}^*V$ *and* V^**, which exist for finite tensor categories.)*
- *For the regular representation* $V = A$ *we have* $M_A = \mathbf{p}\mathbf{s}^t$.
- *If* $\ell = 0$ *then* $K(A) = 0$*, else* $K(A) \cong (\mathbb{Z}/\gamma\mathbb{Z}) \oplus (\mathbb{Z}/d\mathbb{Z})^{\ell-1}$.
- *Assume* $K(V)$ *is finite so that* L_V *has nonzero eigenvalues* $\lambda_1, \ldots, \lambda_\ell$. *Then* $|K(V)| = |\lambda_1 \cdots \lambda_\ell \cdot \gamma/d|$.

Question

What does $\gamma = \gcd(\mathbf{p})$ mean in terms of the structure of A?

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- Let $g_0 = e, g_1, \ldots, g_\ell$ be p-regular conjugacy class representatives.

Critical groups for group representations

- The character values of simple modules (or projective indecomposable modules, resp.) at $g_0, \ldots, g_\ell$ give left (or right, resp.) eigenvectors of M_V with eigenvalues $\chi_V(g_0), \ldots, \chi_V(g_\ell)$.

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- Theorem (Burnside): A tensor-rich V is faithful if $\mathrm{char}(\mathbb{F}) = 0$.

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Theorem (Burciu)

A module V over a Hopf algebra A is the "inflation" of a tensor-rich module over $A / \bigcap_{k \geq 0} \mathrm{Ann}_A(V^{\otimes k})$.

Thank you!