Moments for a generalized coin flip game

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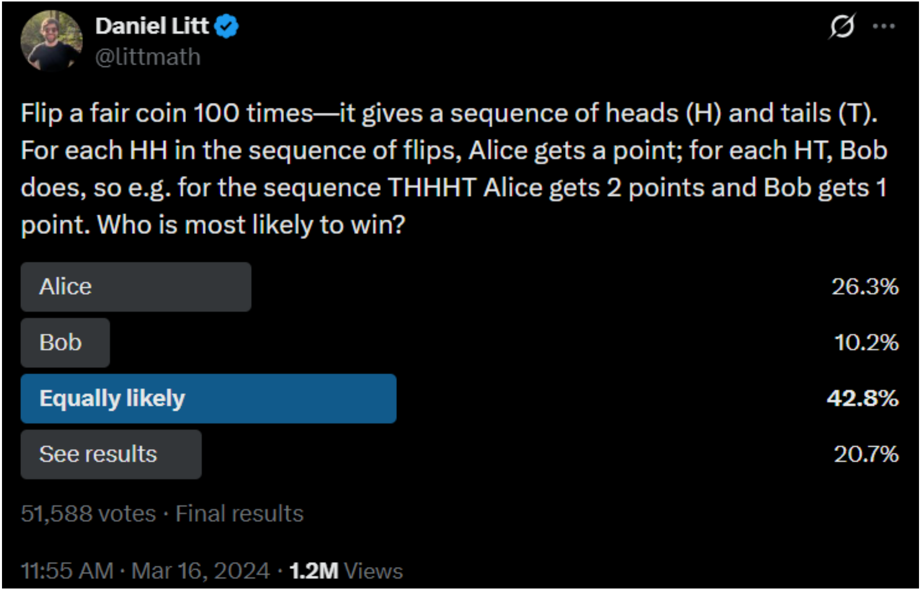
April 2026

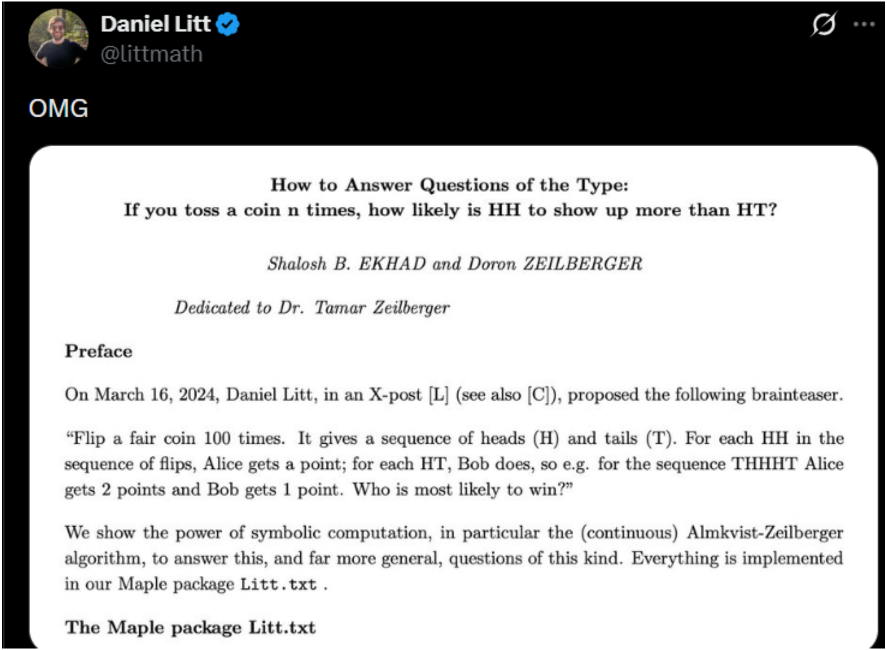

Daniel Litt
@littmath

Flip a fair coin 100 times—it gives a sequence of heads (H) and tails (T). For each HH in the sequence of flips, Alice gets a point; for each HT, Bob does, so e.g. for the sequence THHHT Alice gets 2 points and Bob gets 1 point. Who is most likely to win?

Alice	26.3%
Bob	10.2%
Equally likely	**42.8%**
See results	20.7%

51,588 votes · Final results

11:55 AM · Mar 16, 2024 · **1.2M** Views



How to Answer Questions of the Type:
If you toss a coin n times, how likely is HH to show up more than HT?

Shalosh B. EKHAD and Doron ZEILBERGER

Dedicated to Dr. Tamar Zeilberger

Preface

On March 16, 2024, Daniel Litt, in an X-post [L] (see also [C]), proposed the following brainteaser.

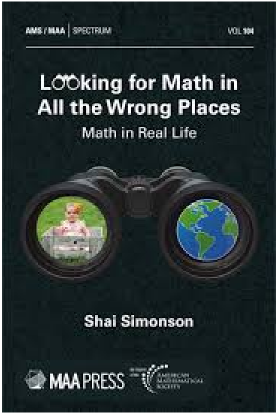
"Flip a fair coin 100 times. It gives a sequence of heads (H) and tails (T). For each HH in the sequence of flips, Alice gets a point; for each HT, Bob does, so e.g. for the sequence THHHT Alice gets 2 points and Bob gets 1 point. Who is most likely to win?"

We show the power of symbolic computation, in particular the (continuous) Almkvist-Zeilberger algorithm, to answer this, and far more general, questions of this kind. Everything is implemented in our Maple package `Litt.txt` .

The Maple package Litt.txt

Introducing the game

- Simonson (2022) discussed a similar game: a carnie offers \$5 to you while in a carnival and asks you to pay \$$n$ back, where n is the number of times you flip a coin until you have two consecutive heads.



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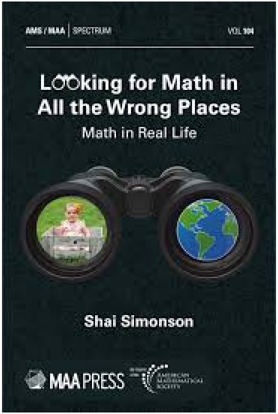
- Although it is nice to start with \$5, on average you will lose \$1 each time when you play this game. Why?



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- It follows that $E = \frac{1}{2}(1+E) + \frac{1}{4}(2+E) + \frac{2}{4}$, so $E = 6$.

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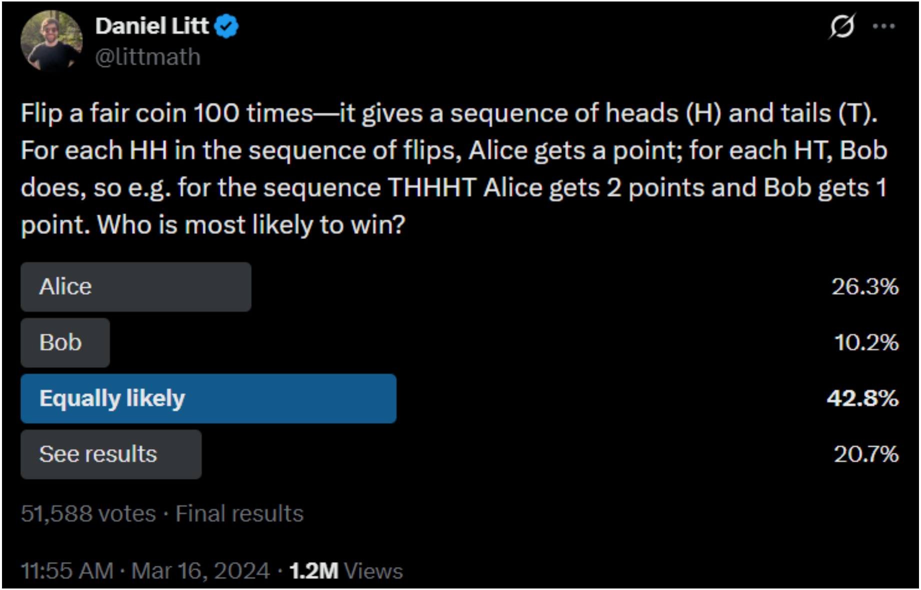
- What if you are offered to switch roles with the carnie but using HT instead of HH to end the game? Don't play since $E = 4$ now!
- Although it takes longer to get HH than HT, it is equally likely to get HT or HH if either occurs, as it is a $\frac{1}{2}$ chance to get H or T after H.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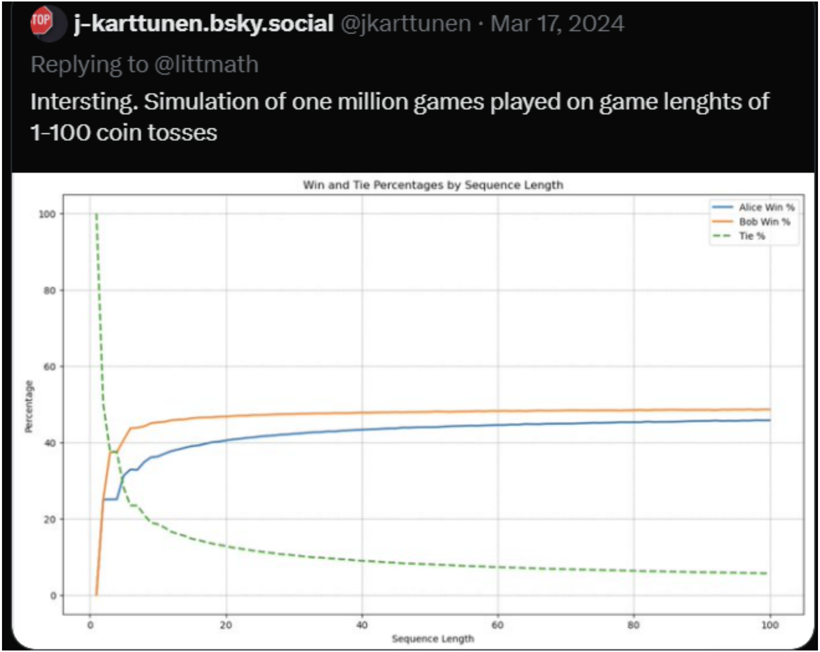
Truth rests with the minority!


j-karttunen.bsky.social @jkarttunen · Mar 17, 2024
Replying to @littmath
Intersting. Simulation of one million games played on game lenghts of 1-100 coin tosses
Win and Tie Percentages by Sequence Length
Alice Win %
Bob Win %
Tie %
Percentage
Sequence Length
0
20
40
60
80
100

Fibonacci number

- The *Fibonacci number* is defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$: $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$.

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- For $n \geq 3$, we have $E_n = E_{n-1} + E_{n-2}$, so $E_n = F_{n-1}$.
- It follows that the expected value of n is

$$\sum_{n=1}^{\infty} \frac{nF_{n-1}}{2^n} = \frac{1 \cdot 0}{2^1} + \frac{2 \cdot 1}{2^2} + \frac{3 \cdot 2}{2^3} + \frac{4 \cdot 3}{2^4} + \frac{5 \cdot 5}{2^5} + \cdots = 6.$$

Higher order Fibonacci numbers

- The *Fibonacci number of order k* is defined by $F_n^k = 0$ for $n < k - 1$, $F_n^k = 1$ for $n = k - 1$, and $F_n^k = F_{n-1}^k + F_{n-2}^k + \cdots + F_{n-k}^k$ for $n \geq k$.

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- We have $F_n^1 = 1$ for $n \geq 0$, $F_n^2 = F_n$ is the Fibonacci number, F_n^3 is the *tribonacci number*, F_n^4 is the *tetranacci number*, and so on.

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- It turns out that $\overline{F}_n^k = F_0^k + F_1^k + \cdots + F_n^k$.

More general results

Theorem (H. 2026+)

Let $E(S)$ be the expected number of flips to obtain a given string S. Then

- $E(\mathrm{H}^k) = 2^{k+1} - 2 = \sum_{n=0}^{\infty} \frac{nF_{n-1}^k}{2^n}$;

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Problem

How to determine $E(S)$ for an arbitrary S?

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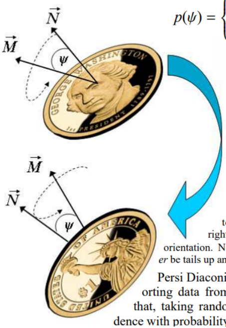
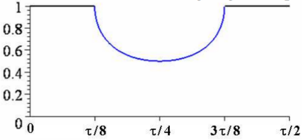
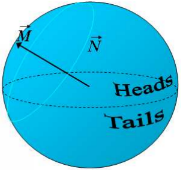
- We can also allow the probabilities of the faces to be distinct.
- Even using a fair coin, the starting side is slightly favored (probability$\approx .51$) by Diaconis, Holmes, and Montgomery (2007).

The Diaconis–Holmes–Montgomery Coin Tossing Theorem *Suppose a coin toss is represented by: ω, the initial angular velocity; t, the flight time; and ψ, the initial angle between the angular momentum vector and the normal to the coin surface, with this surface initially 'heads up'. Consider the pair (ω, t) as a smooth, compactly supported random variable, and let the centre of its distribution tend to infinity in the positive orthant (corresponding to large spin and long flight time). Then with ψ fixed, the limiting probability $p(\psi)$ of the coin landing heads is given by*

$$p(\psi) = \begin{cases} \frac{1}{2} + \frac{2}{\tau}\sin^{-1}(\cot^2\psi) & \text{if } \frac{\tau}{8} < \psi < \frac{3\tau}{8}\ (\tau = 2\pi) \\ 1 & \text{if } 0 < \psi < \frac{\tau}{8} \text{ or } \frac{3\tau}{8} < \psi < \frac{\tau}{2}. \end{cases}$$







The key advance encapsulated in this theorem is the inclusion of precession: the normal $\vec{N}$ to the coin surface rotates about the base of the angular momentum vector $\vec{M}$ as shown on the left. The angle ψ between the two vectors is shown to remain constant. The effect on the orientation of the coin can be viewed in terms of a sphere, above right, whose equator is the flat 'heads up' plane. Only if ψ is sufficiently large will the coin ever flip over into the tails orientation. Note, in particular, that if $\psi = \tau/4$ then the coin spends an equal time heads up and tails up; if $\psi < \tau/8$ it will *never* be tails up and a heads outcome is a certainty. This is in accordance with the function $p(\psi)$ whose graph is plotted above centre.

Persi Diaconis, Susan Holmes and Richard Montgomery published this analysis in 2007, together with supporting data from detailed experiments. Although the heads bias seems conclusively demonstrated they remark that, taking random external factors into consideration: "for tossed coins, the classical assumptions of independence with probability 1/2 are pretty solid."

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Example

$P(\mathrm{H}) = P(\mathrm{T}) = \frac{1}{2}$, $P(\mathrm{HH}) = P(\mathrm{HT}) = \frac{1}{4}$, so $E(\mathrm{HT}) = 4$ since $\mathrm{ov}(\mathrm{HT}) = \{\mathrm{HT}\}$ and $E(\mathrm{HH}) = 4 + 2 = 6$ since $\mathrm{ov}(\mathrm{HH}) = \{\mathrm{H}, \mathrm{HH}\}$.

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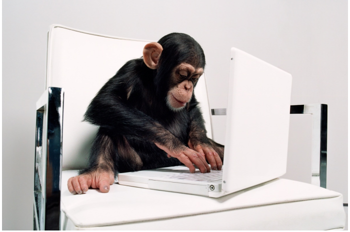
- We also obtain a close formula in terms of set partitions for higher moments using *Faà di Bruno's formula*, a generalization of the chain rule to higher order derivatives.
- The higher moments remain the same when S is reversed.
- Our formula for $E(S)$ provides a natural explanation for some previous results by Janson, Nica, and Segert (2025) on which of two chosen words will occur more frequently when rolling the die many times.

The ABRACADABRA Problem

- If the monkey randomly types a capital letter each time, it will take on average $26^{11} + 26^{4} + 26$ times to produce ABRACADABRA.



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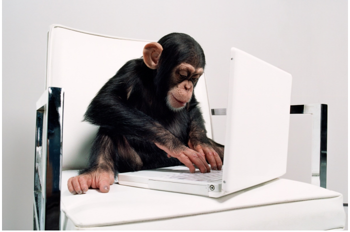

- This answer follows immediately from our formula for $E(S)$ if one uses an unbiased die with $m = 26$ faces and observes that $\mathrm{ov}(\mathrm{ABRACADABRA}) = \{\mathrm{A}, \mathrm{ABRA}, \mathrm{ABRACADABRA}\}$.

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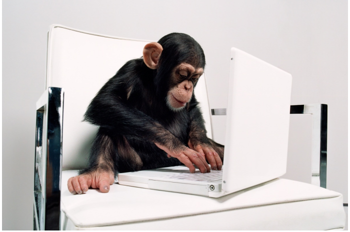

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- The ABRACADABRA problem has been studied using *probability with martingales*. There might be existing work on the higher moments via a probabilistic approach, although we cannot find any.


THANK
YOU