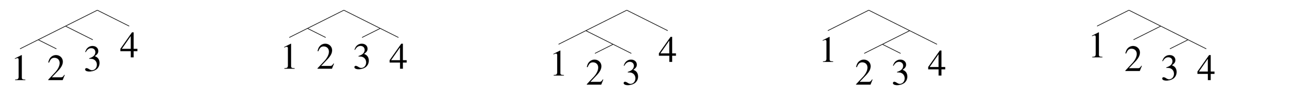
The associative-commutative spectrum of a binary operation

Jia Huang — University of Nebraska at Kearney — huangj2@unk.edu

Erkko Lehtonen — Khalifa University and Universidade Nova de Lisboa — e.lehtonen@fct.unl.pt

Associative spectrum

- A *groupoid* is a set G with a binary operation $*$. Let $\mathcal{P}_*(n)$ be the set of all n-ary term operations on $(G,*)$ induced by bracketings of n variables. Example:



$((x_1*x_2)*x_3)*x_4$ $\quad (x_1*x_2)*(x_3*x_4)$ $\quad (x_1*(x_2*x_3))*x_4$ $\quad x_1*((x_2*x_3)*x_4)$ $\quad x_1*(x_2*(x_3*x_4))$

- We have $1 \le |\mathcal{P}_*(n)| \le C_{n-1}$, where $C_n := \frac{1}{n+1}\binom{2n}{n}$ is the ubiquitous *Catalan number*. The equality $|\mathcal{P}_*(n)| = 1$ holds for all $n \ge 1$ if and only if $*$ is associative. Thus $|\mathcal{P}_*(n)|$ measures the failure of $*$ to be associative.
- Csákány and Waldhauser defined the *associative spectrum* of the binary operation $*$ to be the sequence $s_n^{\mathrm{a}}(*) := |\mathcal{P}_*(n)|$, while Braitt and Silberger called it the *subassociativity type* of the groupoid $(G,*)$. It has been determined for many binary operations.
- It turns out that the associative spectrum has connections with the operad theory. We have a *non-symmetric operad* $\mathcal{P}_* := \{\mathcal{P}_*(n)\}_{n\ge1}$ which has an identity element $1 \in \mathcal{P}_*(1)$ and a composition function satisfying some coherence axioms. The *Hilbert series* of $\mathcal{P}_*$ is $\sum_{n=1}^{\infty}|\mathcal{P}_*(n)|t^n$.

Associative-commutative spectrum

- Let $\overline{\mathcal{P}}_*(n)$ be the set of all n-ary operations induced on $(G,*)$ by the bracketings of all permutations of n variables $x_1,\dots,x_n$. This gives a *symmetric operad* $\overline{\mathcal{P}}_* := \{\overline{\mathcal{P}}_*(n)\}_{n\ge1}$ with Hilbert series $\sum_{n=1}^{\infty}\frac{|\overline{\mathcal{P}}_*(n)|}{n!}t^n$.
- We define the *associative-commutative spectrum* (in brief, *ac-spectrum*) of the binary operation $*$ to be the sequence $s_n^{\mathrm{ac}}(*) := |\overline{\mathcal{P}}_*(n)|$, which measures the nonassociativity and noncommutativity of $*$. It is clear that $s_n^{\mathrm{ac}}(*) \ge 1$; the equality holds for all $n \ge 1$ if and only if $*$ is both commutative and associative.
- For an arbitrary binary operation $*$, we have $s_n^{\mathrm{ac}}(*) \le n!C_{n-1}$, and the equality holds for all $n \ge 1$ when $(G,*)$ is the free groupoid on one generator.
- If $*$ is associative, then $s_n^{\mathrm{ac}}(*) \le n!$, and the equality holds when $(G,*)$ is the free associative groupoid (i.e., the free semigroup) on two generators or any associative noncommutative groupoid with a neutral (i.e., identity) element.

Two-element groupoids

- Every two-element groupoid must be (anti-)isomorphic to $(\{0,1\},*)$ with $x*y$ defined as one of the following: (1) 1, (2) x, (3) $\min\{x,y\}$, (4) $x+y \pmod 2$, (5) $x+1 \pmod 2$, (6) $x \downarrow y$ (negated disjunction, NOR) or (7) $x \to y$ (implication). Csákány and Waldhauser found the associative spectra of all two-element groupoids.
- We have $s_n^{\mathrm{ac}}(*) = 1$ for all $n \ge 1$ if $*$ defined by (1), (3), or (4) since $*$ is both associative and commutative in these three cases.
- The operation $*$ defined by (2) is associative but not commutative, and we have $s_n^{\mathrm{ac}}(*) = n$ for all $n \ge 1$, which is much smaller than the upper bound $n!$ for the ac-spectrum of an associative operation.
- The operation $*$ defined by (5) is neither associative nor commutative, and we have $s_1^{\mathrm{ac}}(*) = 1$, $s_2^{\mathrm{ac}}(*) = 2$, and $s_n^{\mathrm{ac}}(*) = 2n$ for all $n \ge 3$.
- We will discuss the groupoids defined by (6) and (7) later.

Commutative groupoids

- If $(G,*)$ is commutative, then $s_n^{\mathrm{ac}}(*) \le D_{n-1}$, where $D_n := (2n)!/(2^n n!)$ is the number of unordered binary trees with n labeled leaves. This upper bound is achieved when $(G,*)$ is the free commutative groupoid on one generator.
- Given a commutative groupoid $(G,*)$, if $s_n^{\mathrm{ac}}(*) = D_{n-1}$ for all $n \ge 1$, then $*$ must be *totally nonassociative*, i.e., $s_n^{\mathrm{a}}(*) = C_{n-1}$ for all $n \ge 1$.
- The converse of the above does not hold: If $*$ is the arithmetic mean on $\mathbb{R}$ or the geometric/harmonic mean on $\mathbb{R}_+$ then $s_n^{\mathrm{a}}(*) = C_{n-1}$ for all $n \ge 1$ by Csákány and Waldhauser, but we show that $s_n^{\mathrm{ac}}(*)$ equals the number of ways to write 1 as an ordered sum of n powers of 2 for all $n \ge 1$ (OEIS A007178).
- Define $*$ on {rock, paper, scissors} by $x*y = y*x := x$ if x beats y or $x = y$. Then $s_n^{\mathrm{ac}}(*) = D_{n-1}$ and $s_n^{\mathrm{a}}(*) = C_{n-1}$ for all $n \ge 1$.
- Let $(G,*)$ be a commutative groupoid with a neutral element e. Then either
(i) $*$ is associative, in which case $s_n^{\mathrm{a}}(*) = s_n^{\mathrm{ac}}(*) = 1$ for all $n \ge 1$, or
(ii) $s_n^{\mathrm{a}}(*) = C_{n-1}$ and $s_n^{\mathrm{ac}}(*) = D_{n-1}$ for all $n \ge 1$.
- Example: The *Jordan algebra* of $n\times n$ self-adjoint matrices over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ (the algebra of quaternions) with a product defined by $x \circ y := (xy+yx)/2$ is a nonassociative commutative groupoids with a neutral element I_n.

Anticommutative algebras

- An algebra over a field $\mathbb{F}$ of characteristic not 2 is *anticommutative* if it satisfies the identity $xy \approx -yx$, which implies the identity $xx \approx 0$ since $xx \approx -xx$.
- The cross product $\times$ is anticommutative and has a "commutative version" $\bowtie$ defined on $\mathbb{R}^3$ by $\mathbf{i}\bowtie\mathbf{i} = \mathbf{j}\bowtie\mathbf{j} = \mathbf{k}\bowtie\mathbf{k} = 0$, $\mathbf{i}\bowtie\mathbf{j} = \mathbf{k}$, $\mathbf{j}\bowtie\mathbf{k} = \mathbf{i}$, and $\mathbf{k}\bowtie\mathbf{i} = \mathbf{j}$.
- Recently, the first author studied $\bowtie$ in connection with the Norton algebras of certain distance regular graphs. Now we show that the ac-spectrum distinguishes $\times$ and $\bowtie$, although the associative spectrum does not:
(i) $s_n^{\mathrm{ac}}(\bowtie) = D_{n-1}$ for all $n \ge 1$,
(ii) $s_n^{\mathrm{ac}}(\times) = 2D_{n-1}$ for all $n \ge 2$, and
(iii) $s_n^{\mathrm{a}}(\times) = s_n^{\mathrm{a}}(\bowtie) = C_{n-1}$ for all $n \ge 1$.
- A triple (e,f,h) of nonzero elements of a Lie algebra is called an *$\mathfrak{sl}_2$-triple* if $[e,f] = h$, $[h,e] = 2e$, and $[h,f] = -2f$. It is well known that $\mathfrak{sl}_2$-triples exist in every semisimple Lie algebra over a field of characteristic zero.
- Given a Lie algebra over a field of characteristic distinct from 2 with an $\mathfrak{sl}_2$-triple, its Lie bracket $[-,-]$ satisfies $s_n^{\mathrm{ac}}([-,-]) = 2D_{n-1}$ for all $n \ge 2$ and $s_n^{\mathrm{a}}([-,-]) = C_{n-1}$ for all $n \ge 1$.

Some totally nonassociative operations

- Let $(G,*)$ be a groupoid satisfying the identity $(xy)z \approx (xz)y$. Then $s_n^{\mathrm{ac}}(*) \le n^{n-1}$ (the number of unordered rooted trees with n labeled vertices), and if the equality holds for all n, then $s_n^{\mathrm{a}}(*) = C_{n-1}$.
- The exponentiation $a*b := a^b$ for all $a,b \in \mathbb{R}_{\ge0}$ satisfies the above identity and its ac-spectrum reaches the upper bound: $s_n^{\mathrm{ac}}(*) = n^{n-1}$ for all $n \ge 1$.
- For the *implication* $\to$ defined on $\{0,1\}$ by $x \to y := 0$ if $(x,y) = (1,0)$ or $x \to y := 1$ otherwise, we also have $s_n^{\mathrm{ac}}(\to) = n^{n-1}$ for all $n \ge 1$. Hint: use $\leftarrow$.
- The *negated disjunction* $\downarrow$ defined on $\{0,1\}$ by the rule $x \downarrow y = 1$ if and only if $x = y = 0$ is commutative and $s_n^{\mathrm{ac}}(\downarrow) = D_{n-1}$ for all $n \ge 1$.

k-associativity and associative spectrum

- A groupoid $(G,*)$ is *right k-associative* if it satisfies the identity $([x_1x_2\cdots x_{k+1}]_{\mathrm{R}}x_{k+2}) \approx (x_1[x_2\cdots x_{k+2}]_{\mathrm{R}})$, where $[\cdots]_{\mathrm{R}}$ is a shorthand for the rightmost bracketing of the variables occurring between the square brackets.
- Example: $a*b := a + e^{2\pi i/k}b$, which reduces to addition and subtraction if $k = 1,2$. Another example: $f*g := xf+yg$ for all $x,y \in \mathbb{C}[x,y]/(y^k-1)$.
- One can also define the *left k-associativity* similarly. The left or right k-associativity becomes the usual associativity when $k = 1$.
- Previously, Hein and the first author showed that the equivalence relation on binary trees induced by the left k-associativity is the same as the congruence relation on the left depth sequences of binary trees modulo k. The number of equivalence classes is called the *k-modular Catalan number*, which counts many restricted families of Catalan objects and has interesting closed formulas.

k-associativity and associative-commutative spectrum

- We determine the ac-spectra of k-associative binary operations like $a*b := a + e^{2\pi i/k}b$ and $f*g := xf+yg$ for all $x,y \in \mathbb{C}[x,y]/(y^k-1)$.
- If $k = 1$ then we clearly have $s_n^{\mathrm{ac}}(*) = 1$ for all $n \ge 1$. For $k \ge 2$, we have

$$s_n^{\mathrm{ac}}(*) = k!S(n,k) + n\sum_{0\le i\le k-2} i!S(n-1,i), \qquad \forall n \ge 1$$

where the *Stirling number of the second kind* $S(n,k)$ counts partitions of the set $[n] = \{1,2,\dots,n\}$ into k (unordered) blocks.
- When $k = 2$ we have $s_n^{\mathrm{ac}}(*) = 2^n - 2$ for $n \ge 2$ (the n-ary operations obtained by bracketing and permuting $x_1 - x_2 - \cdots - x_n$ are precisely those of the form $\pm x_1 \pm x_2 \cdots \pm x_n$ with at least one plus sign and at least one minus sign).
- When $k = 3$ we have $n\sum_{1\le i\le k-2} i!S(n-1,i) = n$ for all $n \ge 2$.
- When $k = 4$, the sequence $n\sum_{1\le i\le k-2} i!S(n-1,i)$ has simple closed formulas (OEIS A058877) $n2^{n-1} - n = \sum_{1\le j\le n}(n-2+j)2^{n-j-1} = \sum_{1\le j\le n-1}\binom{n}{j}(n-j)$.
- The first author, Mickey, and Xu (also Csákány and Waldhauser) studied the *double minus operation* defined by $a \ominus b := -a-b$ and determined $s_n^{\mathrm{a}}(\ominus)$ (OEIS A000975). Now we show that $s_n^{\mathrm{ac}}(\ominus) = (2^n - (-1)^n)/3$ for all $n \ge 1$, which is the well-known *Jacobsthal sequence* (OEIS A001045).

Remarks and questions

- Csákány and Waldhauser found the associative spectra of some of the 3330 distinct three-element groupoids. What are their ac-spectra?
- Find the ac-spectra of groupoids with properties weaker than associativity, e.g.,
(i) *alternative*: $(x*x)*y \approx x*(x*y)$ and $y*(x*x) = (y*x)*x$ hold;
(ii) *power associative*: any element generates an associative subgroupoid;
(iii) *flexible*: $x*(y*x) \approx (x*y)*x$ holds (e.g., a Lie algebra).
- The Jordan algebra is commutative (hence flexible) and power associative.
- The *Okubo algebra*, which consists of all 3-by-3 trace-zero complex matrices with a product $x \circ y := axy + byx - \mathrm{tr}(xy)I_3/3$ for some $a,b \in \mathbb{C}$ satisfying $a+b = 3ab = 1$, is also flexible and power associative but not alternative.
- The multiplication of octonions is alternative, power associative, and flexible.
- The multiplication of sedenions is power associative and flexible but not alternative.