The binomial transformation

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Binomial Transform

Definition

Given a sequence F, define a (new) sequence G by

$$G_n = \sum_{k=0}^{n} \binom{n}{k} F_k, \quad n \in \mathbf{Z}_{\geq 0}$$

The sequence G is the *binomial transform* of F. Symbolically, we'll write $\mathrm{B}(F)$ for the binomial transform of F.

- $\mathrm{domain}(\mathrm{B}) = \mathrm{codomain}(\mathrm{B}) = \{f \mid f \text{ is a sequence}\}$.
- The earliest mention of binomial transformation I know is in *The Art of Computer Programming*, by Donald Knuth.

(Always) an example

The famous identity $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ translates to

$$\mathrm{B}(n \mapsto 1) = n \mapsto 2^n.$$

Conflating a function with its formula, the result is

$$\mathrm{B}(1)_n = 2^n.$$

- "Conflate" means to combine several concepts into one.

Linearity is almost invariably a clue

For $G = \mathrm{B}(F)$, the first few terms of G are

$$G_0 = \sum_{k=0}^{0} \binom{0}{k} \mathsf{F}_k = \mathsf{F}_0,$$
$$G_1 = \sum_{k=0}^{1} \binom{1}{k} \mathsf{F}_k = \mathsf{F}_0 + \mathsf{F}_1,$$
$$G_2 = \sum_{k=0}^{2} \binom{2}{k} \mathsf{F}_k = \mathsf{F}_0 + 2\,\mathsf{F}_1 + \mathsf{F}_2\,.$$

For $k \in \mathbf{Z}_{\geq 0}$, we have $G_k \in \mathrm{span}\{F_0, F_1, F_2, \ldots, F_k\}$.

Be Nice

Translated to matrix language, $G = \mathrm{B}(F)$ has the nice form

$$\begin{bmatrix} G_0 \\ G_1 \\ G_2 \\ G_3 \\ G_4 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & \ldots \\ 1 & 2 & 1 & 0 & 0 & \ldots \\ 1 & 3 & 3 & 1 & 0 & \ldots \\ 1 & 4 & 6 & 4 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \\ F_4 \\ \vdots \end{bmatrix}.$$

- The size of the matrix is $\infty \times \infty$. That's commonplace.
- The matrix is lower triangular–that makes the sums finite.
- The matrix completely describes B and the other way too. We might as well conflate this matrix with B.

What does a matrix have?

Matrices have transposes. What's the transpose of B applied to F?

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \dots \\ 0 & 1 & 2 & 3 & 4 & \dots \\ 0 & 0 & 1 & 3 & 6 & \dots \\ 0 & 0 & 0 & 1 & 4 & \dots \\ 0 & 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \\ F_4 \\ \vdots \end{bmatrix} = \begin{bmatrix} F_0 + F_1 + F_2 + \cdots \\ F_1 + 2F_2 + 3F_3 + \cdots \\ F_2 + 3F_3 + \cdots \\ F_3 + 4F_4 + \cdots \\ F_4 + \cdots \\ \vdots \end{bmatrix}.$$

In non-matrix language, this defines a transformation B^T defined by

$$\mathrm{B}^T(F) = n \mapsto \sum_{k=n}^{\infty} \binom{k}{n} F_k.$$

- The matrix is upper triangular–the sums are not finite.
- Ouch! We need to be concerned with convergence.

A transformation so nice, we'll do it twice

Definition

For $q \in \mathbf{Z}_{\geq 0}$, (recursively) define

$$\mathrm{B}^{(q)} = \begin{cases} \mathrm{id} & q = 0 \\ \mathrm{B} \circ \mathrm{B}^{(q-1)} & q \in \mathbf{Z}_{\geq 1} \end{cases}.$$

We say $\mathrm{B}^{(q)}$ is the *q-fold composition* of B with itself.

1. By $\circ$, we mean function composition.
2. The function id is the universal identity function:

$$\mathrm{id} = x \in \text{universal set} \mapsto x.$$

3. Thus, $\mathrm{B}^{(0)}(F) = F$, $\mathrm{B}^{(1)}(F) = \mathrm{B}(F)$, $\mathrm{B}^{(2)}(F) = \mathrm{B}(\mathrm{B}(F))$, $\ldots$.

A transformation so nice, we'll do it twice

Theorem (Homework)

For $q \in \mathbf{Z}_{\geq 0}$, we have (assuming $0^0 = 1$)

$$\mathrm{B}^{(q)}(F) = n \mapsto \sum_{k=0}^{n} \binom{n}{k} q^{n-k} F_k.$$

1. The formula for $\mathrm{B}^{(q)}$ extends from $q \in \mathbf{Z}_{\geq 0}$ to $q \in \mathbf{C}$.
2. Extensions like this are nice.

Inversion by q-fold composition

Theorem (Homework)

For $q, r \in \mathbf{C}$, we have $\mathrm{B}^{(q)} \circ \mathrm{B}^{(r)} = \mathrm{B}^{(q+r)}$.

1. $\mathrm{B}^{(q)} \circ \mathrm{B}^{(r)}$ is the composition of $\mathrm{B}^{(q)}$ with $\mathrm{B}^{(r)}$.
2. Traditional notation: Juxtaposition means composition; thus $\mathrm{B}^{(q)} \circ \mathrm{B}^{(r)} = \mathrm{B}^{(q)}\,\mathrm{B}^{(r)}$.
3. $\mathrm{B}^{(0)} = \mathrm{id}$.
4. Thus $\mathrm{B}^{(q)\,-1} = \mathrm{B}^{(-q)}$. Specifically

$$\mathrm{B}^{-1}(F) = n \mapsto \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} F_k.$$

Fame by association

Theorem (Homework)

For $n, k \in \mathbf{Z}_{\geq 0}$, we have

$$\binom{n}{k} = \frac{1}{n!} \mathrm{D}_x^{(n)} \left(\frac{1}{1-x} \right) \left(\frac{x}{1-x} \right)^k \Bigg|_{x \leftarrow 0}$$

1. D is the derivative operator.
2. This theorem implies that the binomial transformation is a member of the (famous) *Riordan group*.
3. Each member of the *Riordan group* is a lower triangular matrix.

Back to basics

For vectors $\mathbf{x}$ and $\mathbf{y}$ and an appropriately sized matrix M, a famous inner product result is

$$\langle \mathbf{x}, M\mathbf{y} \rangle = \langle M^T \mathbf{x}, \mathbf{y} \rangle.$$

If M is nonsingular, we have

$$\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, M^{-1} M\mathbf{y} \rangle = \langle M^{-1\,T} \mathbf{x}, M\mathbf{y} \rangle.$$

In summation (language) and assuming the standard inner product, this is

$$\sum_{k=1}^{n} x_k y_k = \sum_{k=1}^{n} \left(M^{-1\,T} \mathbf{x} \right)_k (M\mathbf{y})_k \, .$$

Back to basics redux

Theorem (Fubini–Tonelli)

Let F and G be sequences and assume that $\sum_{k=0}^{\infty} |F|_k \in \mathbf{R}$ and $\sum_{k=0}^{\infty} |G|_k \in \mathbf{R}$. For $z \in \mathbf{C}$, we have

$$\sum_{k=0}^{\infty} G_k F_k = \sum_{k=0}^{\infty} \mathrm{B}^{(-z)\,T}(G)_k \, \mathrm{B}^{(z)}(F)_k.$$

1. Again, this is a generalization of the well-known linear algebra fact about inner products:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, M^{-1} M \mathbf{y} \rangle = \langle M^{-1\,T} \mathbf{x}, M \mathbf{y} \rangle.$$

Just the facts

For $a, b, c \in \mathbf{C}$ and $d, e \in \mathbf{C} \setminus \mathbf{Z}_{\leq 0}$, define a sequence Q by

$$Q_k = \begin{cases} 1 & k = 0 \\ \frac{(a+k)(b+k)(c+k)}{(d+k)(e+k)} Q_{k-1} & k \in \mathbf{Z}_{\geq 1} \end{cases}.$$

Frobenius (1849–1917) tells us that there is a function ${}_3\mathrm{F}_2$ such that

1. ${}_3\mathrm{F}_2$ is analytic on $\mathbf{C} \setminus [1, \infty)$.
2. for $x \in \mathrm{ball}(0; 1)$, we have

$${}_3\mathrm{F}_2 \left[\begin{matrix} a, b, c \\ d, e \end{matrix} ; x \right] = \sum_{k=0}^{\infty} Q_k x^k.$$

Additionally

1. Frobenius tells us that ${}_3\mathrm{F}_2$ can be extended to a differential function on $\mathbf{C} \setminus [1, \infty)$, but not much else.
2. the function ${}_3\mathrm{F}_2$ has has applications to the dilogarithm function, statistics, the Hahn polynomials, and to the Clebsch–Gordan coefficients.
3. for $x \in \mathrm{ball}(0; 1)$, we have

$$ {}_3\mathrm{F}_2 \left[\begin{matrix} a, b, c \\ d, e \end{matrix} ; x \right] = 1 + \frac{abc}{de} x + \frac{a(a+1)b(b+1)c(c+1)}{2d(d+1)e(e+1)} x^2 + \cdots $$

4. If $k \in \mathbf{Z}_{\geq 0}$, then ${}_3\mathrm{F}_2 \left[\begin{smallmatrix} -k, b, c \\ d, e \end{smallmatrix} ; x \right]$ is an kth degree polynomial in x.

Theorem (BW)

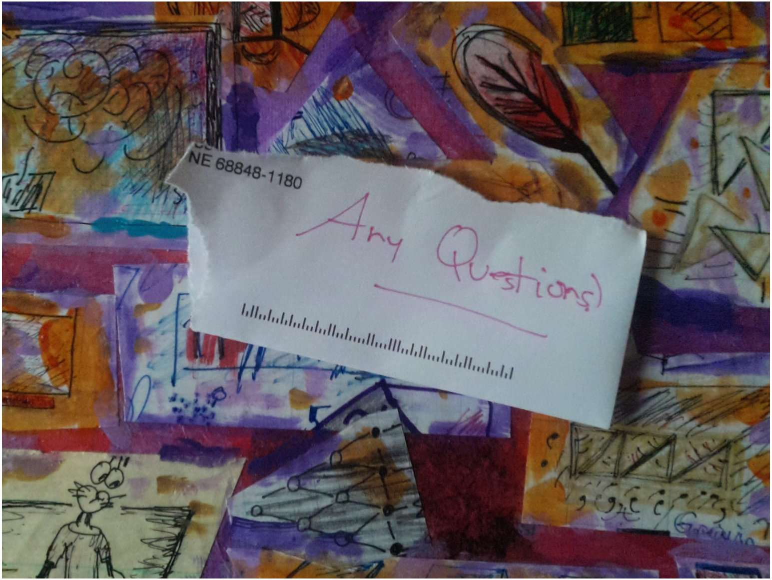
Let $x \in \mathbf{C}_{\neq 0} \setminus [1, \infty)$, and $a, b, c \in \mathbf{C}$ and $d, e \in \mathbf{C} \setminus \mathbf{Z}_{\leq 0}$ Define

$$z = \begin{cases} \frac{x}{2} & x \in \mathbf{R} \\ \frac{|x|^2}{\overline{x}-x} - \frac{x}{\overline{x}-x}\left(1 - |x-1|\right) & x \in \mathbf{C} \setminus \mathbf{R} \end{cases}.$$

Assuming a branch (if any) of $z \in \mathbf{C} \mapsto z^{-a}$ to be $(-\infty, 0]$, a convergent series representation for ${}_3\mathrm{F}_2\left[\begin{smallmatrix} a,b,c \\ d,e \end{smallmatrix}; x\right]$ is

$$(1-z)^{-a} \sum_{k=0}^{\infty} \begin{bmatrix} a \\ 1 \end{bmatrix}_k {}_3\mathrm{F}_2\left[\begin{matrix} -k, b, c \\ d, e \end{matrix}; x/z\right] \left(\frac{z}{z-1}\right)^k.$$

1. Proof: the binomial transformation.
2. $\left[\begin{smallmatrix} a \\ 1 \end{smallmatrix}\right]_k = \prod_{\ell=0}^{k-1} \frac{a+\ell}{1+\ell}$.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Any Questions)