Tiling the integers

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We can tile the set $\mathbb{Z}$ of all integers with translates of $S = \{0, 1\}$. 

Coven and Meyerowitz (1999) generalized the result of Newman to the case $|S| = p^a q^b$ for two primes $p, q$. 

Example: $S = \{0, 1, 2\}$ works but $S = \{0, 1, 3\}$ does not work. 

Example: $S = \{1, 4, 8, 13\}$ works but $S = \{1, 4, 8, 15\}$ does not work. 

Observation: We may assume $0 \in S$, without loss of generality. 

If we can tile $\mathbb{Z}$ with translates of $S$, the density of the tiling is $1/|S|$. 

Example: Tiling $\mathbb{Z}$ with translates of $S = \{0, 1, 2\}$ has a density $1/3$. 

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Tiling the Integers without overlaps

- We can tile the set $\mathbb{Z}$ of all integers with translates of $S = \{0, 1\}$.
- When can we tile $\mathbb{Z}$ with translates of a finite set $S \subseteq \mathbb{Z}$?

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If we can tile $\mathbb{Z}$ with translates of $S$, the **density** of the tiling is $1/|S|$.

Example: Tiling $\mathbb{Z}$ with translates of $S = \{0, 1, 2\}$ has a density $1/3$. 
We have to allow overlaps to tile \( \mathbb{Z} \) with translates of \( S = \{0, 1, 3\} \).
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The smallest density of such a tiling is $\gamma(\mathbb{Z}, S) = 2/5 > 1/3$. 

Theorem (H. 2019+): If $k$ is a positive integer then $\gamma(\mathbb{Z}, \{0, 1, 3k+2\}) = 1/3$. 

Observation: If $s \mid t$ then $\gamma(\mathbb{Z}, \{0, s, t\}) = \gamma(\mathbb{Z}, \{0, 1, t/s\})$. 

Problem: Determine $\gamma(\mathbb{Z}, \{0, s, t\})$ with $s \not\mid t$ (e.g., $S = \{0, 2, 3\}$).
We have to allow overlaps to tile \( \mathbb{Z} \) with translates of \( S = \{0, 1, 3\} \).

The smallest density of such a tiling is \( \gamma(\mathbb{Z}, S) = \frac{2}{5} > \frac{1}{3} \).

If \( k \) is an integer then \( \gamma(\mathbb{Z}, \{0, 1, 3k + 2\}) = \frac{1}{3} \).
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Theorem (H. 2019+): If $k$ is a positive integer then

$$\gamma(\mathbb{Z}, \{0, 1, 3k\}) = \gamma(\mathbb{Z}, \{0, 1, -3k + 1\}) = \frac{2k}{6k - 1},$$

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Tiling the integers with overlaps

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Observation: If \( s \mid t \) then \( \gamma(\mathbb{Z}, \{0, s, t\}) = \gamma(\mathbb{Z}, \{0, 1, t/s\}) \).

Problem: Determine \( \gamma(\mathbb{Z}, \{0, s, t\}) \) with \( s \nmid t \) (e.g., \( S = \{0, 2, 3\} \)).
Thank you!