Nonassociativity of some binary operations

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Nonassociativity of binary operations

Fact

- *Let $*$ be a binary operation on a set X.*
- *Let $x_0, x_1, \ldots, x_n$ be X-valued indeterminates.*
- *If $*$ is associative then the expression $x_0 * x_1 * \cdots * x_n$ is unambiguous.*
- *If $*$ is nonassociative then $x_0 * x_1 * \cdots * x_n$ depends on parentheses.*
- *The number of ways to parenthesize $x_0 * x_1 * \cdots * x_n$ is the Catalan number $C_n := \frac{1}{n+1}\binom{2n}{n}$, e.g., $(C_n)_{n=0}^{6} = (1,1,2,5,14,42,132)$.*

Example (Subtraction, $n = 3$)

$$\left.\begin{array}{r}
((x_0-x_1)-x_2)-x_3 = x_0 - x_1 - x_2 - x_3 \\
(x_0-x_1)-(x_2-x_3) = x_0 - x_1 - x_2 + x_3 \\
(x_0-(x_1-x_2))-x_3 = x_0 - x_1 + x_2 - x_3 \\
x_0-((x_1-x_2)-x_3) = x_0 - x_1 + x_2 + x_3 \\
x_0-(x_1-(x_2-x_3)) = x_0 - x_1 + x_2 - x_3
\end{array}\right\} \Rightarrow \left\{\begin{array}{l}
C_3 = 5 \\
C_{-,3} = 4 \\
\widetilde{C}_{-,3} = 2
\end{array}\right.$$

Nonassocitivity measurements

Definition

- Parenthesizations of $x_0 * x_1 * \cdots * x_n$ are *equivalent* if they give the same function from X^{n+1} to X. Call this *$(*, n)$-equivalence relation*.
- Define $C_{*,n}$ to be the number of $(*, n)$-equivalence classes.
- Define $\widetilde{C}_{*,n}$ to be the largest size of $(*, n)$-equivalence classes.

Observation

- *In general,* $1 \leq C_{*,n} \leq C_n$ *and* $1 \leq \widetilde{C}_{*,n} \leq C_n$.
- $C_{*,n} = 1, \forall n \geq 0 \Leftrightarrow *$ *is associative* $\Leftrightarrow \widetilde{C}_{*,n} = C_n, \forall n \geq 0$.
- *Thus* $C_{*,n}$ *and* $\widetilde{C}_{*,n}$ *measure how far* $*$ *is away from being associative.*

Problem

Determine $C_{*,n}$ *and* $\widetilde{C}_{*,n}$ *for a given binary operation* $*$.

Binary trees

Remark

N. J. Lord (1987) introduced the *depth of nonassociativity* of a binary operation $*$, which can now be written as

$$\inf\{n+1 : C_{*,n} < C_n\} = \inf\{n+1 : \widetilde{C}_{*,n} > 1\}.$$

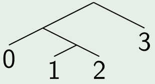
Fact

Parenthesizations of $x_0 * x_1 * \cdots * x_n$

$$\updownarrow$$

(full) binary trees with $n+1$ *leaves*

Example



$(x_0*(x_1*x_2))*x_3$

Definition

- Let $\mathcal{T}_n$ denote the set of all binary trees with $n+1$ leaves.
- Define the *$(*, n)$-relation* on $\mathcal{T}_n$: write $t \sim_* t'$ if $t, t' \in \mathcal{T}_n$ correspond to equivalent parenthesizations of $x_0 * x_1 * \cdots * x_n$.

A generalization of associativity

Definition

- A binary operation $*$ is *k-associative* if
$$(x_0 * \cdots * x_k) * x_{k+1} = x_0 * (x_1 * \cdots * x_{k+1})$$
where the operations in parentheses are performed left to right.
- Suppose the $(*, n)$-relation $\sim_*$ is generated by k-associativity.
- Write $C_{k,n} := C_{*,n}$ (*k-modular Catalan number*) and $\widetilde{C}_{k,n} := \widetilde{C}_{*,n}$

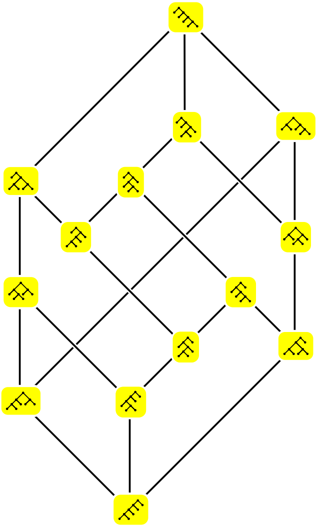
Example (Generalization of "$+$" ($k = 1$) and "$-$" ($k = 2$))

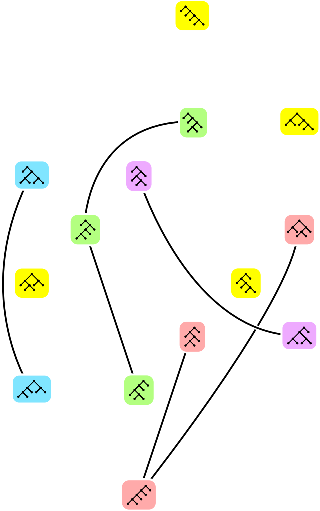
Let $\omega := e^{2\pi i/k}$ be a primitive kth root of unity. Then $*$ is k-associative if
$$a * b := \omega a + b, \quad \forall a, b \in \mathbb{C}.$$

Observation ($k = 1$: Tamari order)

The $(*, n)$-equivalence classes are components of $\mathcal{T}_n$ under k-associative order generated by k-rotation: $(t_0 \wedge \cdots \wedge t_k) \wedge t_{k+1} \leftrightarrow t_0 \wedge (t_1 \wedge \cdots \wedge t_{k+1})$.

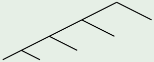
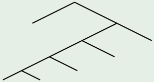
Tamari order and 2-associative order on $\mathcal{T}_4$





Components of k-associative order

Example (comb_4 and comb_4^1)

Theorem (Hein and H.)

- *A binary tree is maximal (or minimal) in the k-associative order if and only if it avoids the binary tree comb_{k+1} (or comb_k^1) as a subtree.*
- *Each component in k-associative order has a unique minimal tree.*

Definition

The *left depth* $\delta_i(t)$ (or *right depth* $\rho_i(t)$) of leaf i in $t \in \mathcal{T}_n$ is the number of edges to the left (right) in the unique path from the root of t down to i.

Theorem (Hein and H.)

$t \sim_* t'$ if and only if $\delta_i(t) \equiv \delta_i(t') \pmod{k}$ for all i.

Connections to other objects

Fact

There are well-known bijections among many families of Catalan objects.

Proposition (Hein and H.)

For $n \geq 0$ and $k \geq 1$, $C_{k,n}$ enumerates the following:

1. *the set of binary trees with $n+1$ leaves avoiding comb_k^1,*
2. *plane trees with n non-root nodes, each of degree less than k,*
3. *Dyck paths of length $2n$ avoiding DU^k (a down-step immediately followed by k up-steps),*
4. *partitions bounded by $(n-1, n-2, \ldots, 1, 0)$ with each positive part occurring fewer than k times,*
5. *$2 \times n$ standard Young tableaux which contain no list of k consecutive numbers in the top row other than $1, 2, \ldots, \ell$ for any $\ell \in [n]$,*
6. *permutations of $[n]$ avoiding 1-3-2 and $23\cdots(k+1)1$.*

Formulas for $C_{k,n}$ and $\widetilde{C}_{k,n}$

Theorem (Hein and H.)

For $k, n \geq 1$, we have

$$C_{k,n} = \sum_{\substack{\lambda \subseteq (k-1)^n \\ |\lambda| < n}} \frac{n - |\lambda|}{n} m_\lambda(1^n) = \sum_{0 \leq j \leq (n-1)/k} \frac{(-1)^j}{n} \binom{n}{j} \binom{2n - jk}{n+1},$$

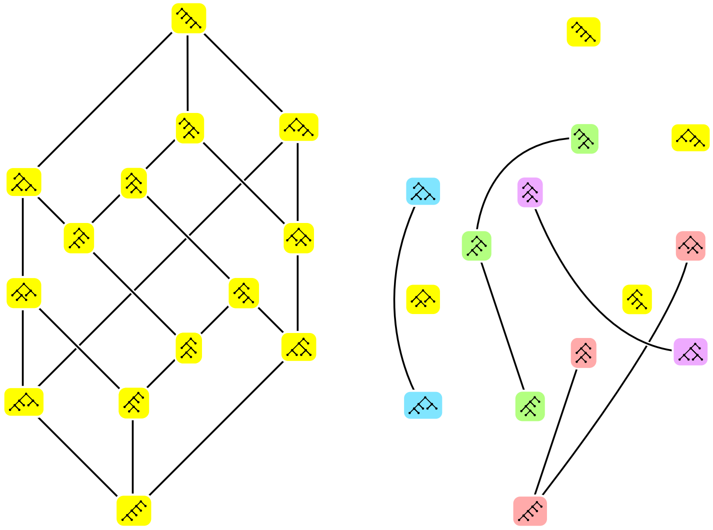
$$\widetilde{C}_{k,n} = \sum_{0 \leq j \leq n/k} \frac{n - jk}{n} \binom{n+j-1}{j}.$$

Moreover, the number of components of $\mathcal{T}_n$ in k-associative order with size $\widetilde{C}_{k,n}$ is C_m, where m is the least positive integer congruent to n modulo k.

Proof.

One proof uses generating functions and Lagrange inversion. The other proof is more direct, using Dyck paths (and sign-reversing involutions). □

Tamari order and 2-associative order on $\mathcal{T}_4$



Modular Catalan numbers

Example ($C_{k,n}$ for $n \le 10$ and $k \le 8$)

n	0	1	2	3	4	5	6	7	8	9	10	
$C_{1,n}$	1	1	1	1	1	1	1	1	1	1	1	A000012
$C_{2,n}$	1	1	2	4	8	16	32	64	128	256	512	A011782
$C_{3,n}$	1	1	2	5	13	35	96	267	750	2123	6046	A005773
$C_{4,n}$	1	1	2	5	14	41	124	384	1210	3865	12482	A159772
$C_{5,n}$	1	1	2	5	14	42	131	420	1375	4576	15431	new
$C_{6,n}$	1	1	2	5	14	42	132	428	1420	4796	16432	new
$C_{7,n}$	1	1	2	5	14	42	132	429	1429	4851	16718	new
$C_{8,n}$	1	1	2	5	14	42	132	429	1430	4861	16784	new
C_n	1	1	2	5	14	42	132	429	1430	4862	16796	A000108

Question

- $\lim_{n\to\infty} C_{n+1}/C_n = 4$, $\lim_{n\to\infty} C_{k,n+1}/C_{k,n} = ?$
- *There is a formula* $C_{3,n} = \sum_{0\le i\le n-1} \binom{n-1}{i}\binom{i}{\lfloor i/2\rfloor}$ *obtained by*
  - *Gouyou-Beauchamps and Viennot in studies of directed animals, and*
  - *Panyushev using affine Weyl group of the Lie algebra* $\mathfrak{sp}_{2n}$ *or* $\mathfrak{so}_{2n+1}$.

Is there a generalization of this formula from $k = 3$ *to* $k \ge 4$*?*

A generalization

Example

Define $a * b := \omega a + \eta b$ for $a, b \in \mathbb{C}$, where $\omega := e^{2\pi i/k}$ and $\eta := \omega^{2\pi i/\ell}$.

Definition

- An element ω of finite order in a semigroup satisfies $\omega^{d+k} = \omega^d$; the smallest d and k are the called *index* and *period* of ω.
- Define $a * b := \omega a + \eta b$ for all a, b in a ring R, where ω (or η) is a fixed element of R with index d (or e) and period k (or ℓ).

Problem (Work in progress with undergraduates)

- *Characterize the $(*, n)$-relation $\sim_*$ on $\mathcal{T}_n$ for the above defined $*$?*
- *Compute the nonassociativity measurements $C_{*,n}$ and $\widetilde{C}_{*,n}$?*

Remark

The elements ω and η may be not independent.

Double Minus

Definition

- Define $a * b := \omega a + \eta b$ for $a, b \in \mathbb{C}$, where $\omega := e^{2\pi i/k}$ and $\eta := \omega^{2\pi i/\ell}$. If $k = \ell = 2$ we have $a \ominus b := -a - b$.
- Let $C_{\ominus,n,r}$ be the number of distinct results from $x_0 \ominus x_1 \ominus \cdots \ominus x_n$ with exactly r plus signs. Let $C_{\ominus,n} := \sum_{0 \le r \le n+1} C_{\ominus,n,r}$.

Theorem (H., Mickey, and Xu)

- *If $n \ge 1$ and $0 \le r \le n+1$ then*

$$C_{\ominus,n,r} = \begin{cases} \binom{n+1}{r}, & \text{if } n+r \equiv 1 \pmod 3 \text{ and } n \neq 2r-2, \\ \binom{n+1}{r} - 1, & \text{if } n+r \equiv 1 \pmod 3 \text{ and } n = 2r-2, \\ 0, & \text{if } n+r \not\equiv 1 \pmod 3. \end{cases}$$

- *For $n \ge 1$ we have* $C_{\ominus,n} = \begin{cases} \frac{2^{n+1}-1}{3}, & \text{if } n \text{ is odd;} \\ \frac{2^{n+1}-2}{3}, & \text{if } n \text{ is even.} \end{cases}$

A truncated/modified Pascal Triangle

Example ($C_{\ominus,n,r}$ for $n \le 10$ and $0 \le r \le n+1$)

r	0	1	2	3	4	5	6	7	8	9	10	11
$C_{\ominus,0,r}$		1										
$C_{\ominus,1,r}$	1											
$C_{\ominus,2,r}$			2									
$C_{\ominus,3,r}$		4			1							
$C_{\ominus,4,r}$	1			9								
$C_{\ominus,5,r}$			15			6						
$C_{\ominus,6,r}$		7			34			1				
$C_{\ominus,7,r}$	1			56			28					
$C_{\ominus,8,r}$			36			125			9			
$C_{\ominus,9,r}$		10			210			120			1	
$C_{\ominus,10,r}$	1			165			461			55		

Double Minus

Definition

- Define $a * b := \omega a + \eta b$ for $a, b \in \mathbb{C}$, where $\omega := e^{2\pi i/k}$ and $\eta := \omega^{2\pi i/\ell}$. If $k = \ell = 2$ we have $a \ominus b := -a - b$.
- Let $C_{\ominus,n,r}$ be the number of distinct results from $x_0 \ominus x_1 \ominus \cdots \ominus x_n$ with exactly r plus signs. Let $C_{\ominus,n} := \sum_{0 \le r \le n+1} C_{\ominus,n,r}$.

Theorem (H., Mickey, and Xu)

- *If $n \ge 1$ and $0 \le r \le n+1$ then*

$$C_{\ominus,n,r} = \begin{cases} \binom{n+1}{r}, & \text{if } n + r \equiv 1 \pmod 3 \text{ and } n \neq 2r-2, \\ \binom{n+1}{r} - 1, & \text{if } n + r \equiv 1 \pmod 3 \text{ and } n = 2r-2, \\ 0, & \text{if } n + r \not\equiv 1 \pmod 3. \end{cases}$$

- *For $n \ge 1$ we have* $C_{\ominus,n} = \begin{cases} \frac{2^{n+1}-1}{3}, & \text{if } n \text{ is odd;} \\ \frac{2^{n+1}-2}{3}, & \text{if } n \text{ is even.} \end{cases}$

OEIS A000975

Definition

The sequence A000975 $(A_n : n \geq 1) = (1, 2, 5, 10, 21, 42, 85, \ldots)$ has many equivalent characterizations, such as the following.

- $A_1 = 1$, $A_{n+1} = 2A_n$ if n is odd, and $A_{n+1} = 2A_n + 1$ if n is even.
- A_n is the integer with an alternating binary representation of length n. $(1 = 1_2,\ 2 = 10_2,\ 5 = 101_2,\ 10 = 1010_2,\ 21 = 10101_2,\ \ldots)$
- $A_n = \left\lfloor \frac{2^{n+1}}{3} \right\rfloor = \frac{2^{n+2}-3-(-1)^n}{6} = \begin{cases} \frac{2^{n+1}-1}{3}, & \text{if } n \text{ is odd;} \\ \frac{2^{n+1}-2}{3}, & \text{if } n \text{ is even.} \end{cases}$
- A_n is the number of moves to solve the n-ring Chinese Rings puzzle. $n = 4$: 0000-0001-0011-0010-0110-0111-0101-0100-1100-1101-1111

Question

Are there natural bijections between distinct results from parenthesizing $x_0 \ominus x_1 \ominus \cdots \ominus x_n$ and any other family of objects enumerated by A_n?

Another Generalization

Definition

Define a binary operation $*$ on $\mathbb{C}[x,y]/(x^{d+k}-x^d, y^{e+\ell}-y^e)$ by

$$f * g := xf + yg$$

Observation

- *A parenthesization of $f_0 * \cdots * f_n$ corresponding to $t \in \mathcal{T}_n$ equals*
$$x^{\delta_0(t)}y^{\rho_0(t)}f_0 + \cdots + x^{\delta_n(t)}y^{\rho_n(t)}f_n.$$
- *It follows that $t \sim_* t'$ if and only if $\delta(t) \sim_k^d \delta(t')$ and $\rho(t) \sim_\ell^e \rho(t')$.*

Definition

Given two integer sequences $\mathbf{a} = (a_0, \ldots, a_n)$ and $\mathbf{b} = (b_0, \ldots, b_n)$, define

- $\mathbf{a} \sim_k \mathbf{b}$ if $a_i \equiv b_i \pmod{k}$ for $i = 0, \ldots, n$,
- $\mathbf{a} \sim^d \mathbf{b}$ if either $a_i = b_i$ or $\min\{a_i, b_i\} \geq d$ for $i = 0, \ldots, n$, and
- $\mathbf{b} \sim_k^d \mathbf{c}$ if $\mathbf{a} \sim_k \mathbf{b}$ and $\mathbf{a} \sim^d \mathbf{b}$.

The case $k = \ell = 1$

Definition

- Suppose $t \sim_* t'$ if and only if $\delta(t) \sim^d \delta(t')$ and $\rho(t) \sim^e \rho(t')$.
- Let $C_n^{d,e} := C_{*,n}$ and $\widetilde{C}_n^{d,e} := \widetilde{C}_{*,n}$.

Theorem (Hein and H.)

- *The relation "$\sim_*$" is generated by 1-rotations at left depth $\geq d-1$ and right depth $\geq e-1$. This can be viewed as associativity of left depth d and right depth e.*
- *If $0 \leq n < d+e$ then $\widetilde{C}^{d,e} = 1$. If $n \geq d+e$ then $\widetilde{C}^{d,e} = C_{n+2-d-e}$ and the number of $(*, n)$-classes with this largest size is $\binom{d+e-2}{e-1}$.*
- *The generating function $C^{d,e}(x) := \sum_{n \geq 0} C^{d,e} x^{n+1}$ satisfies*

$$C^{d,e}(x) = x + C^{d-1,e}(x) C^{d,e-1}(x)$$

where $d = 0$ or $e = 0$ is treated as $d = 1$ or $e = 1$.

The case $k = \ell = 1$ and $e \le 2$

Fact

- $\{C_n^{d,1} : d \ge 1, n \ge 0\}$ *is* A080934 *in OEIS which enumerates*
  1. *permutations in* $\mathfrak{S}_n$ *avoiding both* 1-3-2 *and* 1-2-3-$\cdots$-d *(Kitaev–Remmel–Tiefenbruck),*
  2. *plane trees with* n *nodes, all of depth at most* d*,*
  3. *binary trees with* $n+1$ *leaves, all of left depth at most* d*,*
  4. *Dyck paths of length* $2n$ *with height at most* d*.*
- $C_n^{2,2}$ *(OEIS* A045623*) enumerates weak compositions of* $n-1$ *with exactly one zero and is the binomial transform of* $1, 1, 2, 2, 3, 3, \ldots$*.*
- $C_n^{3,2}$ *(OEIS* A142586*) is the binomial transform of* $\lfloor (\frac{1+\sqrt{5}}{2})^n \rfloor$*.*

Proposition (?)

Let $F_0(x) := 0$, $F_1(x) := 1$, *and* $F_n(x) := F_{n-1}(x) + xF_{n-2}(x)$*. Then*

$$C^{d,1}(x) = \frac{xF_{d+1}(-x)}{F_{d+2}(-x)} \quad \text{and} \quad C^{d,2}(x) = \frac{xF_4(-x)F_{d+1}(-x) + x^{d+2}}{F_4(-x)F_{d+2}(-x)}.$$

The case $e = \ell = 1$

Definition

- Suppose $t \sim_* t'$ if and only if $\delta(t) \sim_k^d \delta(t')$. This can be viewed as *left k-associativity of left depth d*.
- Let $C_{k,n}^d := C_{*,n}$ and $\widetilde{C}_{k,n}^d := \widetilde{C}_{*,n}$.

Theorem (Hein and H.)

- *For $k, d \geq 1$ and $n \geq 0$, the number $C_{k,n}^d$ enumerates*
  1. *binary trees in $\mathcal{T}_n$ avoiding comb_k^1 at any left depth $\geq d-1$,*
  2. *plane trees with multi-degree $(d_0, \ldots, d_n)$ satisfying $d_0 + \cdots + d_{i-1} - i \geq d \Rightarrow d_i < k$, $\forall i \in [n]$,*
  3. *Dyck paths of length $2n$ avoiding DU^k at height at least d,*
- *For $m, n, d \geq 0$ and $k \geq 1$, $C_k^d(x) := \sum_{n\geq 0} C_k^d x^{n+1}$ satisfies*

$$C_k^{d+1}(x) = x/(1 - C_k^d(x)).$$

The case $e = \ell = 1$ and $k \le 3$ or $d \le 2$

Proposition (?)

We have $C_1^d(x) = C^{d,1}(x)$, $C_2^d(x) = C^{d+1,1}(x)$, *and*

$$C_3^d(x) = \frac{xG_d(-x) + x^2 F_{d-1}(-x)\sqrt{1-2x-3x^2}}{G_{d+1}(-x) + xF_d(-x)\sqrt{1-2x-3x^2}}$$

where $G_0(x) := 1 + x$, $G_1(x) := 2x^2$, *and* $G_n(x) := G_{n-1}(x) + xG_{n-2}(x)$. *See OEIS A005773 and A054391–A054394 for* $\{C_{3,n}^d\}$, $1 \le d \le 5$.

Theorem (Hein and H.)

$$\begin{aligned} C_{k,n}^2(x) &= 1 + \sum_{1 \le i \le n-1} \frac{i}{n-i} \sum_{0 \le j \le (n-i-1)/k} (-1)^j \binom{n-i}{j} \binom{2n-i-jk-1}{n} \\ &= 1 + \sum_{1 \le i \le n-1} \sum_{\lambda \subseteq (k-1)^{n-i}} \frac{n-i-|\lambda|}{n-i} \binom{n-|\lambda|-1}{n-|\lambda|-i} m_\lambda(1^{n-i}). \end{aligned}$$

Questions

Conjecture

- *Let $\mathcal{U}_n$ be the algebra of n-by-n upper triangular matrices over a field. Then nilpotent (two-sided) ideals of order at most d in $\mathcal{U}_n$ is $C_n^{d,1}$.*
  - *L. Shapiro: commutative ideals of $\mathcal{U}_n$ are enumerated by $2^{n-1} = C_n^{2,1}$.*
  - *Nilpotent ideals of order ≤ 2 in $\mathcal{U}_n$ are exactly those commutative.*
- *For $k, \ell \geq 1$ and $n \geq 0$, $C_{k,\ell,n} = C_{k+\ell-1,1,n}$.*

Question

- *Let $f * g := xf + gy$ for all $f, g \in \mathbb{C}[x,y]/I$.*
- *How to determine $C_{k,\ell,n}^{d,e} := C_{*,n}$ and $\widetilde{C}_{k,\ell,n}^{d,e} := \widetilde{C}_{*,n}$ when $I = (x^{d+k} - x^d, y^{e+\ell} - y^e)$ with arbitrary $d, e, k, \ell \geq 1$?*
- *What are $C_{*,n}$ and $\widetilde{C}_{*,n}$ for other ideals I, such as $I = ((x^k - 1)y^d)$?*

Question (Total nonassociativity)

When do we have $C_{*,n} = C_n$, or equivalently $\widetilde{C}_{*,n} = 1$?