# Combinatorics of non-associative binary operations

Jia Huang

University of Nebraska at Kearney
*E-mail address*: huangj2@unk.edu

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This is joint work with Nickolas Hein (Benedictine College),
Madison Mickey (UNK) and Jianbai Xu (UNK)

# Nonassociativity of binary operations

- Let $*$ be a binary operation on a set $X$. Let $x_0, x_1, \ldots, x_n$ be $X$-valued indeterminates.
- If $*$ is associative then the expression $x_0 * x_1 * \cdots * x_n$ is unambiguous. Example: $x_0 + x_1 + \cdots + x_n$.
- If $*$ is nonassociative then $x_0 * x_1 * \cdots * x_n$ depends on parentheses.

$$
\begin{aligned}
((x_0 - x_1) - x_2) - x_3 &= x_0 - x_1 - x_2 - x_3 \\
(x_0 - x_1) - (x_2 - x_3) &= x_0 - x_1 - x_2 + x_3 \\
(x_0 - (x_1 - x_2)) - x_3 &= {\color{red} x_0 - x_1 + x_2 - x_3} \\
x_0 - ((x_1 - x_2) - x_3) &= x_0 - x_1 + x_2 + x_3 \\
x_0 - (x_1 - (x_2 - x_3)) &= {\color{red} x_0 - x_1 + x_2 - x_3}
\end{aligned}
$$

- The number of ways to parenthesize $x_0 * x_1 * \cdots * x_n$ is the *Catalan number* $C_n := \frac{1}{n+1}\binom{2n}{n}$, e.g., $(C_n)_{n=0}^{6} = (1, 1, 2, 5, 14, 42, 132)$.
- How many distinct results can be obtained from $x_0 * x_1 * \cdots * x_n$?

# Nonassocitivity measurements

- Parenthesizations of $x_0 * x_1 * \cdots * x_n$ are *equivalent* if they give the same function from $X^{n+1}$ to $X$. Call this *$(*, n)$-equivalence relation*.
- Define $C_{*,n}$ to be the number of $(*, n)$-equivalence classes.
- Define $\widetilde{C}_{*,n}$ to be the largest size of $(*, n)$-equivalence classes.

$$\left.\begin{array}{r}
((x_0-x_1)-x_2)-x_3 = x_0 - x_1 - x_2 - x_3 \\
(x_0-x_1)-(x_2-x_3) = x_0 - x_1 - x_2 + x_3 \\
(x_0-(x_1-x_2))-x_3 = x_0 - x_1 + x_2 - x_3 \\
x_0-((x_1-x_2)-x_3) = x_0 - x_1 + x_2 + x_3 \\
x_0-(x_1-(x_2-x_3)) = x_0 - x_1 + x_2 - x_3
\end{array}\right\} \Rightarrow \begin{cases} C_3 = 5 \\ C_{-,3} = 4 \\ \widetilde{C}_{-,3} = 2 \end{cases}$$

## Observation

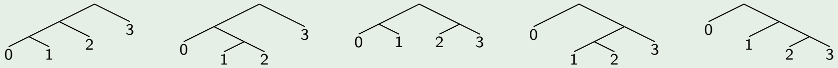
- *In general,* $1 \le C_{*,n} \le C_n$ *and* $1 \le \widetilde{C}_{*,n} \le C_n$.
- $C_{*,n} = 1, \forall n \ge 0 \Leftrightarrow *$ *is associative* $\Leftrightarrow \widetilde{C}_{*,n} = C_n, \forall n \ge 0$ .
- *Thus* $C_{*,n}$ *and* $\widetilde{C}_{*,n}$ *measure how far* $*$ *is away from being associative.*

# Binary trees

## Fact

*Parenthesizations of* $x_0 * x_1 * \cdots * x_n \leftrightarrow$ *(full) binary trees with* $n+1$ *leaves*

## Example




| | | | | |
|---|---|---|---|---|
| $\updownarrow$ | $\updownarrow$ | $\updownarrow$ | $\updownarrow$ | $\updownarrow$ |
| $((x_0 * x_1) * x_2) * x_3$ | $(x_0 * (x_1 * x_2)) * x_3$ | $(x_0 * x_1) * (x_2 * x_3)$ | $x_0 * ((x_1 * x_2) * x_3)$ | $x_0 * (x_1 * (x_2 * x_3))$ |
| $\delta = (3, 2, 1, 0)$ | $\delta = (2, 2, 1, 0)$ | $\delta = (2, 1, 1, 0)$ | $\delta = (1, 2, 1, 0)$ | $\delta = (1, 1, 1, 0)$ |
| $\rho = (0, 1, 1, 1)$ | $\rho = (0, 1, 2, 1)$ | $\rho = (0, 1, 1, 2)$ | $\rho = (0, 1, 2, 2)$ | $\rho = (0, 1, 2, 3)$ |

## Definition

The *left depth* $\delta_i(t)$ (or *right depth* $\rho_i(t)$) of leaf $i$ in $t \in \mathcal{T}_n$ is the number of edges to the left (right) in the unique path from the root of $t$ down to $i$.

# A generalization of associativity

## Definition

- A binary operation $*$ is *$k$-associative* if

$$(x_0 * \cdots * x_k) * x_{k+1} = x_0 * (x_1 * \cdots * x_{k+1})$$

where the operations in parentheses are performed left to right.
- Write $C_{k,n} := C_{*,n}$ (*$k$-modular Catalan number*) and $\widetilde{C}_{k,n} := \widetilde{C}_{*,n}$ for any operation $*$ satisfying precisely the $k$-associativity.

## Example (Generalization of "$+$" ($k=1$) and "$-$" ($k=2$))

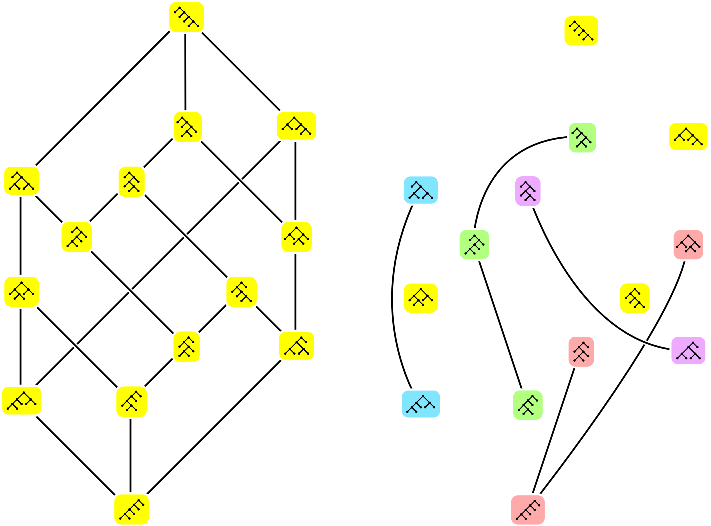
Let $\omega := e^{2\pi i/k}$ be a primitive $k$th root of unity. Then $*$ is $k$-associative if

$$a * b := \omega a + b, \quad \forall a, b \in \mathbb{C}.$$

## Observation ($k = 1$: Tamari order)

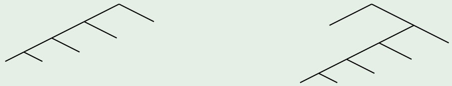
*The $k$-associativity gives the $k$-associative order on binary trees.*

# Tamari order and 2-associative order on $\mathcal{T}_4$



# Components of $k$-associative order

## Example ($\mathrm{comb}_4$ and $\mathrm{comb}_4^1$)



## Theorem (Hein and H. 2017)

- *A binary tree is maximal (or minimal) in the $k$-associative order if and only if it avoids the binary tree $\mathrm{comb}_{k+1}$ (or $\mathrm{comb}_k^1$) as a subtree.*
- *Each component in $k$-associative order has a unique minimal tree.*

## Theorem (Hein and H. 2017)

*Two binary trees $t$ and $t'$ correspond to equivalent parenthesizations if and only if $\delta_i(t) \equiv \delta_i(t') \pmod{k}$ for all $i$.*

# Connections to other objects

## Fact

*There are well-known bijections among many families of Catalan objects.*

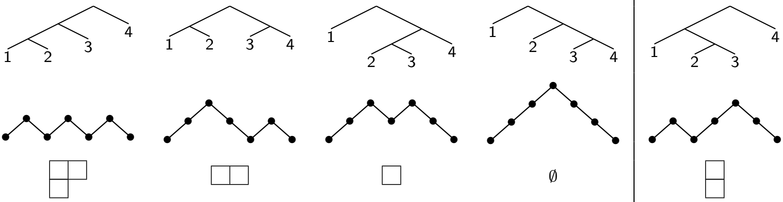
## Proposition (Hein and H. 2017)

*For $n \geq 0$ and $k \geq 1$, $C_{k,n}$ enumerates the following:*

1. *the set of binary trees with $n+1$ leaves avoiding $\mathrm{comb}_k^1$,*
2. *plane trees with $n$ non-root nodes, each of degree less than $k$,*
3. *Dyck paths of length $2n$ avoiding $DU^k$ (a down-step immediately followed by $k$ up-steps),*
4. *partitions bounded by $(n-1, n-2, \ldots, 1, 0)$ with each positive part occurring fewer than $k$ times,*
5. *$2 \times n$ standard Young tableaux which contain no list of $k$ consecutive numbers in the top row other than $1, 2, \ldots, \ell$ for any $\ell \in [n]$,*
6. *permutations of $[n]$ avoiding 1-3-2 and $23\cdots(k+1)1$.*

# Examples of Catalan objects

| | | | | |
|---|---|---|---|---|
| $((x_1*x_2)*x_3)*x_4$ | $(x_1*x_2)*(x_3*x_4)$ | $x_1*((x_2*x_3)*x_4)$ | $x_1*(x_2*(x_3*x_4))$ | $(x_1*(x_2*x_3))*x_4$ |




| | | | | |
|---|---|---|---|---|
| 1 3 5 / 2 4 6 | 1 2 5 / 3 4 6 | 1 2 4 / 3 5 6 | 1 2 3 / 4 5 6 | 1 3 4 / 2 5 6 |
| 321 | 312 | 213 | 123 | 231 |

# Formulas for $C_{k,n}$ and $\widetilde{C}_{k,n}$

## Theorem (Hein and H. 2017)

*For $k, n \geq 1$, we have*

$$C_{k,n} = \sum_{\substack{\lambda \subseteq (k-1)^n \\ |\lambda| < n}} \frac{n - |\lambda|}{n} m_\lambda(1^n) = \sum_{0 \leq j \leq (n-1)/k} \frac{(-1)^j}{n} \binom{n}{j} \binom{2n - jk}{n+1},$$

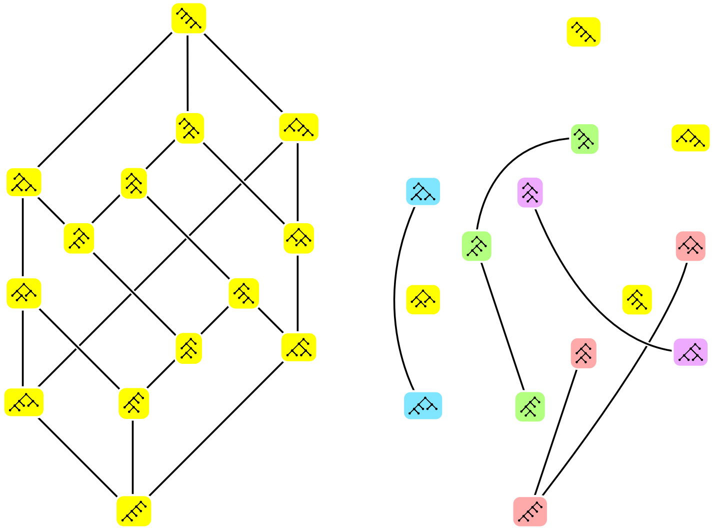
$$\widetilde{C}_{k,n} = \sum_{0 \leq j \leq n/k} \frac{n - jk}{n} \binom{n + j - 1}{j}.$$

*Moreover, the number of components in $k$-associative order with size $\widetilde{C}_{k,n}$ is $C_m$, where $m$ is the least positive integer congruent to $n$ modulo $k$.*

## Proof.

One proof uses generating functions and Lagrange inversion. The other proof is more direct, using Dyck paths (and sign-reversing involutions). □

# Tamari order and 2-associative order on $\mathcal{T}_4$



# Modular Catalan numbers

## Example ($C_{k,n}$ for $n \leq 10$ and $k \leq 8$)

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $C_{1,n}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | A000012 |
| $C_{2,n}$ | 1 | 1 | 2 | 4 | 8 | 16 | 32 | 64 | 128 | 256 | 512 | A011782 |
| $C_{3,n}$ | 1 | 1 | 2 | 5 | 13 | 35 | 96 | 267 | 750 | 2123 | 6046 | A005773 |
| $C_{4,n}$ | 1 | 1 | 2 | 5 | 14 | 41 | 124 | 384 | 1210 | 3865 | 12482 | A159772 |
| $C_{5,n}$ | 1 | 1 | 2 | 5 | 14 | 42 | 131 | 420 | 1375 | 4576 | 15431 | new |
| $C_{6,n}$ | 1 | 1 | 2 | 5 | 14 | 42 | 132 | 428 | 1420 | 4796 | 16432 | new |
| $C_{7,n}$ | 1 | 1 | 2 | 5 | 14 | 42 | 132 | 429 | 1429 | 4851 | 16718 | new |
| $C_{8,n}$ | 1 | 1 | 2 | 5 | 14 | 42 | 132 | 429 | 1430 | 4861 | 16784 | new |
| $C_n$ | 1 | 1 | 2 | 5 | 14 | 42 | 132 | 429 | 1430 | 4862 | 16796 | A000108 |

## Question

- $\lim_{n\to\infty} C_{n+1}/C_n = 4$, $\lim_{n\to\infty} C_{k,n+1}/C_{k,n} =?$
- *There is a formula* $C_{3,n} = \sum_{0\leq i\leq n-1} \binom{n-1}{i}\binom{i}{\lfloor i/2\rfloor}$ *obtained by*
  - *Gouyou-Beauchamps and Viennot in studies of directed animals, and*
  - *Panyushev using affine Weyl group of the Lie algebra* $\mathfrak{sp}_{2n}$ *or* $\mathfrak{so}_{2n+1}$.

*Is there a generalization of this formula from* $k = 3$ *to* $k \geq 4$*?*

# Double Minus

## Definition

- Define $a * b := \omega a + \eta b$ for $a, b \in \mathbb{C}$, where $\omega := e^{2\pi i/k}$ and $\eta := \omega^{2\pi i/\ell}$. If $k = \ell = 2$ we have $a \ominus b := -a - b$.
- Let $C_{\ominus,n,r}$ be the number of distinct results from $x_0 \ominus x_1 \ominus \cdots \ominus x_n$ with exactly $r$ plus signs. Let $C_{\ominus,n} := \sum_{0 \le r \le n+1} C_{\ominus,n,r}$.

## Theorem (H., Mickey, and Xu 2017)

- *If $n \geq 1$ and $0 \leq r \leq n+1$ then*

$$C_{\ominus,n,r} = \begin{cases} \binom{n+1}{r}, & \text{if } n+r \equiv 1 \pmod 3 \text{ and } n \neq 2r-2, \\ \binom{n+1}{r} - 1, & \text{if } n+r \equiv 1 \pmod 3 \text{ and } n = 2r-2, \\ 0, & \text{if } n+r \not\equiv 1 \pmod 3. \end{cases}$$

- *For $n \geq 1$ we have* $C_{\ominus,n} = \begin{cases} \frac{2^{n+1}-1}{3}, & \text{if } n \text{ is odd;} \\ \frac{2^{n+1}-2}{3}, & \text{if } n \text{ is even.} \end{cases}$

# A truncated/modified Pascal Triangle

## Example ($C_{\ominus,n,r}$ for $n \leq 10$ and $0 \leq r \leq n+1$)

| $r$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $C_{\ominus,0,r}$ | | 1 | | | | | | | | | | |
| $C_{\ominus,1,r}$ | 1 | | | | | | | | | | | |
| $C_{\ominus,2,r}$ | | | 2 | | | | | | | | | |
| $C_{\ominus,3,r}$ | | 4 | | | 1 | | | | | | | |
| $C_{\ominus,4,r}$ | 1 | | | 9 | | | | | | | | |
| $C_{\ominus,5,r}$ | | | 15 | | | 6 | | | | | | |
| $C_{\ominus,6,r}$ | | 7 | | | 34 | | | 1 | | | | |
| $C_{\ominus,7,r}$ | 1 | | | 56 | | | 28 | | | | | |
| $C_{\ominus,8,r}$ | | | 36 | | | 125 | | | 9 | | | |
| $C_{\ominus,9,r}$ | | 10 | | | 210 | | | 120 | | | 1 | |
| $C_{\ominus,10,r}$ | 1 | | | 165 | | | 461 | | | 55 | | |

# Double Minus

## Definition

- Define $a * b := \omega a + \eta b$ for $a, b \in \mathbb{C}$, where $\omega := e^{2\pi i/k}$ and $\eta := \omega^{2\pi i/\ell}$. If $k = \ell = 2$ we have $a \ominus b := -a - b$.
- Let $C_{\ominus,n,r}$ be the number of distinct results from $x_0 \ominus x_1 \ominus \cdots \ominus x_n$ with exactly $r$ plus signs. Let $C_{\ominus,n} := \sum_{0\le r\le n+1} C_{\ominus,n,r}$.

## Theorem (H., Mickey, and Xu 2017)

- *If $n \geq 1$ and $0 \leq r \leq n+1$ then*

$$C_{\ominus,n,r} = \begin{cases} \binom{n+1}{r}, & \text{if } n+r \equiv 1 \pmod 3 \text{ and } n \neq 2r-2, \\ \binom{n+1}{r} - 1, & \text{if } n+r \equiv 1 \pmod 3 \text{ and } n = 2r-2, \\ 0, & \text{if } n+r \not\equiv 1 \pmod 3. \end{cases}$$

- *For $n \geq 1$ we have* $C_{\ominus,n} = \begin{cases} \frac{2^{n+1}-1}{3}, & \text{if } n \text{ is odd;} \\ \frac{2^{n+1}-2}{3}, & \text{if } n \text{ is even.} \end{cases}$

# OEIS A000975

## Definition

The sequence A000975 $(A_n : n \geq 1) = (1, 2, 5, 10, 21, 42, 85, \ldots)$ has many equivalent characterizations, such as the following.

- $A_1 = 1$, $A_{n+1} = 2A_n$ if $n$ is odd, and $A_{n+1} = 2A_n + 1$ if $n$ is even.
- $A_n$ is the integer with an alternating binary representation of length $n$. $(1 = 1_2,\ 2 = 10_2,\ 5 = 101_2,\ 10 = 1010_2,\ 21 = 10101_2,\ \ldots)$
- $A_n = \left\lfloor \frac{2^{n+1}}{3} \right\rfloor = \frac{2^{n+2}-3-(-1)^n}{6} = \begin{cases} \frac{2^{n+1}-1}{3}, & \text{if } n \text{ is odd;} \\ \frac{2^{n+1}-2}{3}, & \text{if } n \text{ is even.} \end{cases}$
- $A_n$ is the number of moves to solve the $n$-ring Chinese Rings puzzle. $n = 4$: 0000-0001-0011-0010-0110-0111-0101-0100-1100-1101-1111

## Question

*Are there natural bijections between distinct results from parenthesizing $x_0 \ominus x_1 \ominus \cdots \ominus x_n$ and any other family of objects enumerated by $A_n$?*

# Another Generalization

## Definition

Let $C_{k,\ell,n}^{d,e} := C_{*,n}$ be the number of distinct results obtained from parenthesizing $x_0 * x_1 * \cdots * x_n$, where $*$ is defined as

$$f * g := xf + yg, \quad \forall f, g \in \mathbb{C}[x,y]/(x^{d+k} - x^d, y^{e+\ell} - y^e)$$

## Observation

*A parenthesization of $f_0 * \cdots * f_n$ corresponding to $t \in \mathcal{T}_n$ equals*

$$x^{\delta_0(t)} y^{\rho_0(t)} f_0 + \cdots + x^{\delta_n(t)} y^{\rho_n(t)} f_n.$$

*So one can study $C_{k,\ell,n}^{d,e}$ using the leaf depths in binary trees.*

## Remark

We have results on $C_{k,\ell,n}^{d,e}$ when two or three of the parameters $k, \ell, d, e$ are set to be one. In the remainder of this talk, we focus on $C_{1,1,n}^{d,1}$ and discuss its connections with the algebra of upper triangular matrices.

# Ideals of upper triangular matrices

## Definition

- Let $\mathcal{U}_n$ be the algebra of all $n$-by-$n$ upper triangular matrices

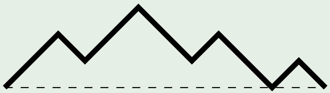
$$\begin{pmatrix} * & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ 0 & 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & * \end{pmatrix}$$

  where a star $*$ is an arbitrary entry from a fixed field $\mathbb{F}$ (e.g., $\mathbb{R}$).
- A (two-sided) ideal $I$ of $\mathcal{U}_n$ is a vector subspace of $\mathcal{U}_n$ such that $XI \subseteq I$ and $IX \subseteq I$ for all $X \in \mathcal{U}_n$.
- A ideal $I$ is nilpotent if $I^k = 0$ for some $k \geq 1$. The smallest $k$ such that $I^k = 0$ is the (nilpotent) order of $I$.
- A ideal $I$ of $\mathcal{U}_n$ is commutative if $AB = BA$ for all $A, B \in I$.

# Nilpotent ideals

## Example (A nilpotent ideal of $\mathcal{U}_6$ and its corresponding Dyck path)

$$I = \begin{bmatrix} 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$



height $= 3$

## Observation

- *An nilpotent ideal of $\mathcal{U}_n$ is represented by a matrix of $0$'s and $*$'s separated by a Dyck path of length $2n$.*
- *The number of such ideals is the Catalan number $C_n := \frac{1}{n+1}\binom{2n}{n}$.*
- *The number of all ideals of $\mathcal{U}_n$ is the Catalan number $C_{n+1}$.*

# Commutative ideals

**Proposition (L. Shapiro, 1975)**

*The number of commutative ideals of* $\mathcal{U}_n$ *is* $2^{n-1}$.

**Problem**

*Find a direct proof of the above result.*

**Example (What is a direct proof?)**

The number of subsets of $\{1, 2, \ldots, n\}$ is $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$.
This can be proved directly by considering if a subset contains $i$ for each $i$.

**Observation**

*An ideal of* $\mathcal{U}_n$ *is commutative if and only if it has nilpotent order* 1 *or* 2.

**Definition**

Let $C_n^d$ be the number of nilpotent ideals of $\mathcal{U}_n$ with order at most $d$.

# Nilpotent order

## Observation

*The order of a nilpotent ideal $I$ of $\mathcal{U}_n$ is the largest possible length $d$ of an admissible sequence, that is, a sequence $(i_1, i_2, \ldots, i_d)$ such that the entry $(i_j, i_{j+1})$ is a star $*$ in the matrix form of $I$ for all $j = 1, 2, \ldots, d-1$.*

## Example

The following ideal has nilpotent order is 4 since the sequence $(1, 3, 5, 6)$ is admissible and there is no longer admissible sequence.

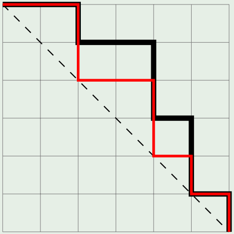
$$I = \begin{bmatrix} 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

# Bounce Paths

## Observation

*Let $I$ be an ideal of $\mathcal{U}_n$ corresponding to a Dyck path $D$. Then the nilpotent order of $I$ is the number of times the bounce path of $D$ bounces off the main diagonal.*

## Example (Bounce Path)



- The bounce path has 4 bounces.
- The Dyck path $D$ has height 3.

## Fact (Andrews–Krattenthaler–Orsina–Papi 2002, Haglund 2008)

*Bijection $\zeta$ : Dyck paths with height $d \leftrightarrow$ Dyck paths with $d$ bounces.*

# Generalization of Commutative Ideals

**Theorem (H.-Rhoades)**

*Dyck paths of length $2n$ with height at most $d$ are counted by $C_n^d$. Hence $C_n^d$ is the sequence A080934 in OEIS and interpolates between $1$ and $C_n$.*

**Example**

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | n |
|---|---|---|---|---|---|---|---|---|
| $C_n^1$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $C_n^2$ | 1 | 2 | 4 | 8 | 16 | 32 | 64 | $2^{n-1}$ |
| $C_n^3$ | 1 | 2 | 5 | 13 | 34 | 89 | 233 | $F_{2n-1}$ |
| $C_n^4$ | 1 | 2 | 5 | 14 | 41 | 122 | 365 | $\frac{1}{2}(1+3^{n-1})$ |
| $C_n$ | 1 | 2 | 5 | 14 | 42 | 132 | 429 | $\frac{1}{n+1}\binom{2n}{n}$ |

# Ideals of Lie Algebras

## Definition

- Let $\mathfrak{sl}_n(\mathbb{C})$ be the (type A semisimple) Lie algebra of all $n \times n$ complex matrices with zero trace under the Lie bracket $[X, Y] := XY - YX$.
- Let $\mathfrak{b}$ be the Borel subalgebra of upper triangular matrices of $\mathfrak{sl}_n(\mathbb{C})$.

## Theorem (Andrews–Krattenthaler–Orsina–Papi 2002)

*The number of ad-nilpotent ideals of $\mathfrak{b}$ with order at most $d-1$ is $C_n^d$.*

## Problem

- *Find a natural order-preserving bijection between nilpotent ideals of $\mathcal{U}_n$ and ad-nilpotent ideals of $\mathfrak{b}$. (The exponential map?)*
- *The above theorem has been generalized from type A to other types [Krattenthaler–Orsina–Papi 2002]. Is there a similar generalization for nilpotent ideals of $\mathcal{U}_n$?*

# Generating function

**Definition**

- Let $C^d(x) := \sum_{n\geq 0} C_n^d x^{n+1}$ for $d \geq 1$, and let $C^0(x) := x$.
- Let $F_i(x) := i$ for $i = 0, 1$, and $F_n(x) := F_{n-1}(x) - xF_{n-2}(x)$, $n \geq 2$.

**Proposition (de Bruijn–Knuth–Rice 1972)**

*For $n \geq 1$ we have $F_n(x) = \sum_{0\leq i\leq (n-1)/2} \binom{n-1-i}{i}(-x)^i$.*

**Proposition (Kreweras 1970)**

*For $d \geq 1$ we have* $$C^d(x) = \frac{x}{1 - C^{d-1}(x)} = \frac{xF_{d+1}(x)}{F_{d+2}(x)}.$$

**Example**

$C^1(x) = \frac{x}{1-x}$, $C^2(x) = \frac{x}{1-\frac{x}{1-x}} = \frac{x(1-x)}{1-2x}$, $C^3(x) = \frac{x}{1-\frac{x}{1-\frac{x}{1-x}}} = \frac{x(1-2x)}{1-3x+x^2}$

# Closed Formulas for $C_n^d$

**Theorem (Andrews–Krattenthaler–Orsina–Papi 2002)**

*The number $C_n^d$ has the following closed formulas:*

$$C_n^d = \sum_{i\in\mathbb{Z}} \frac{2i(d+2)+1}{2n+1}\binom{2n+1}{n-i(d+2)} = \det\left[\binom{i-j+d}{j-i+1}\right]_{i,j=1}^{n-1}$$

$$= \sum_{0=i_0\le i_1\le\cdots\le i_{d-1}\le i_d=n} \prod_{0\le j\le d-2} \binom{i_{j+2}-i_j-1}{i_{j+1}-i_j}.$$

**Theorem (de Bruijn–Knuth–Rice 1972)**

*The number of plane trees with $n+1$ nodes of depth at most $d$ is*

$$C_n^d = \frac{2^{2n+1}}{d+2} \sum_{1\le j\le d+1} \sin^2(j\pi/(d+2))\cos^{2n}(j\pi/(d+2)).$$

# More on the number $C_n^d$

## Theorem (Hein and H.)

*For $n, d \geq 1$ we have $C_{1,1,n}^{d,1} = C_n^d$.*

## Definition

A composition of $n$ is a sequence $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ of positive integers such that $\alpha_1 + \cdots + \alpha_\ell = n$. Let $\max(\alpha) := \max\{\alpha_1, \ldots, \alpha_\ell\}$ and $\ell(\alpha) = \ell$.

## Proposition (Hein and H.)

*For $n, d \geq 1$ we have*

$$C_n^d = \sum_{\substack{\alpha \models n \\ \max(\alpha) \leq (d+1)/2}} (-1)^{n-\ell(\alpha)} \binom{d-\alpha_1}{\alpha_1 - 1} \prod_{2 \leq i \leq \ell(\alpha)} \binom{d+1-\alpha_i}{\alpha_i}$$

Thank you!