Modular Catalan Numbers

Jia Huang (joint with Nick Hein)

University of Nebraska at Kearney
E-mail address: huangj2@unk.edu

September 25, 2015

Outline

- Introduce the *modular Catalan numbers* from parenthesizations
- Use (full) binary trees to study the modular Catalan numbers.
- Give connections between the modular Catalan number and other Catalan objects.
- Investigate closed formulas and generating functions of the modular Catalan numbers.
- Questions for future study.

Binary Operations and Parentheses

Definition

A *binary operation* $*$ defined on a set X is a function $X \times X \to X$ sending $(a, b) \in X \times X$ to $a * b \in X$. (Examples: $+$ and $-$.)

Observation

The expression $a * b * c$ *is ambiguous and depends on parentheses:* $(a * b) * c$ *and* $a * (b * c)$ *are not equal in general.*

Example (Addition and Subtraction)

Addition is *associtiave*: $(a + b) + c = a + (b + c)$.
Subtraction is NOT associative:

$$(a - b) - c = a - b - c$$
$$a - (b - c) = a - b + c$$

Catalan number

Fact

In general, the number of ways to parenthesize $x_1 * x_2 * \cdots * x_{n+1}$ *is the* *Catalan number* $C_n := \frac{1}{n+1}\binom{2n}{n}$. $(1, 1, 2, 5, 14, 42, ...)$

Example (Addition)

Since addition is associtiave, $x_1 + \cdots + x_{n+1}$ is unambiguous for any $n \geq 0$.

Example (Subtraction, $n = 3$)

$$\begin{aligned}
((a-b)-c)-d &= a - b - c - d \\
(a-b)-(c-d) &= a - b - c + d \\
(a-(b-c))-d &= a - b + c - d \\
a-((b-c)-d) &= a - b + c + d \\
a-(b-(c-d)) &= a - b + c - d
\end{aligned}$$

Modular Catalan number

Observation

Given a binary operation $*$ on a set X, each parentheization of $x_1 * \cdots * x_{n+1}$ gives a function from X^{n+1} to X.

Fact

- *Denote by $C_{n,*}$ the number of distinct functions obtained from parenthesizations of $x_1 * \cdots * x_{n+1}$. Then we have $1 \leq C_{n,*} \leq C_n$.*
- *For example, $C_{n,+} = 1$, and $C_{n,-} = 2^{n-1}$.*

Definition

- Define a binary operation $a \circledk b := a + \omega b$ where $\omega = e^{2\pi i/k}$ is a primitive kth root of unity. For example, ① is $+$ and ② is $-$.
- The *(k-)modular Catalan number* is $C_{n,k} := C_{n,\textcircled{k}}$.

Basic Properties

n	0	1	2	3	4	5	6	7	8	9	10	11	12	OEIS
$C_{n,1}$	1	1	1	1	1	1	1	1	1	1	1	1	1	A000012
$C_{n,2}$	1	1	2	4	8	16	32	64	128	256	512	1024	2048	A000079
$C_{n,3}$	1	1	2	5	13	35	96	267	750	2123	6046	17303	49721	A005773
$C_{n,4}$	1	1	2	5	14	41	124	384	1210	3865	12482	40677	133572	A159772
$C_{n,5}$	1	1	2	5	14	42	131	420	1375	4576	15431	52603	180957	new
$C_{n,6}$	1	1	2	5	14	42	132	428	1420	4796	16432	56966	199444	new
$C_{n,7}$	1	1	2	5	14	42	132	429	1429	4851	16718	58331	205632	new
$C_{n,8}$	1	1	2	5	14	42	132	429	1430	4861	16784	58695	207452	new
C_n	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	A000108

Fact

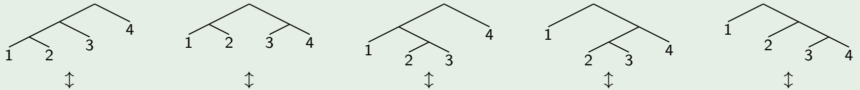
- $C_{n,1} = 1$ *and* $C_{n,2} = 2^{n-1}$ *for* $n \geq 0$.
- $C_{0,k} = C_{1,k} = 1$ *for* $k \geq 1$.
- $C_{n,k} = C_n$ *for* $n \leq k$.
- $C_{k+1,k} = C_{k+1} - 1$ *for* $k \geq 1$.
- $C_{k+2,k} = C_{k+2} - k - 4$ *for* $k \geq 2$.
- $C_{k+3,k} = C_{k+3} - (k^2 + 11k + 30)/2$ *for* $k \geq 3$.

Binary trees

Observation

Parenthesizations of $x_1 * \cdots * x_{n+1} \leftrightarrow$ *to binary trees with* $n+1$ *leaves.*

Example



$((x_1-x_2)-x_3)-x_4$	$(x_1-x_2)-(x_3-x_4)$	$(x_1-(x_2-x_3))-x_4$	$x_1-((x_2-x_3)-x_4)$	$x_1-(x_2-(x_3-x_4))$
$\Vert$	$\Vert$	$\Vert$	$\Vert$	$\Vert$
$x_1-x_2-x_3-x_4$	$x_1-x_2-x_3+x_4$	$x_1-x_2+x_3-x_4$	$x_1-x_2+x_3+x_4$	$x_1-x_2+x_3-x_4$
$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$
$(0,1,1,1)$	$(0,1,1,2)$	$(0,1,2,1)$	$(0,1,2,2)$	$(0,1,2,3)$

Definition

The *skew depth* $\mathrm{d}_i(t)$ of the ith leaf in a binary tree t is the number of steps to the right in the unique downward path from the root of t to i.

Observation

A parenthesization of $x_1 \circledk x_2 \circledk \cdots \circledk x_{n+1}$ *is completely determined by the skew depth* $\mathrm{d}(t) = (\mathrm{d}_1, \ldots, \mathrm{d}_{n+1})$ *of the corresponding tree* t *modulo* k*, since this parenthesization may be written as*

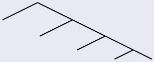
$$\omega^{\mathrm{d}_1} x_1 + \omega^{\mathrm{d}_2} x_2 + \omega^{\mathrm{d}_3} x_3 + \cdots + \omega^{\mathrm{d}_{n+1}} x_{n+1}.$$

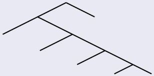
Definition

We say two binary trees with $n+1$ leaves are *k-equivalent* if their skew depths are congruent modulo k.

Theorem (Hein and H.)

Each k-equivalence class of binary trees contains exactly one tree avoiding comb_k^1*. For example,* comb_4 *and* comb_4^1 *are given below.*





Modular Catalan Objects

Theorem (Hein and H.)

The modular Catalan number $C_{n,k}$ enumerates the following objects:

1. *binary trees with $n+1$ leaves avoiding comb_k^1 as a subtree,*
2. *plane trees with $n{+}1$ nodes whose non-root nodes have degree less than k,*
3. *Dyck paths of length $2n$ avoiding DU^k (a down-step followed immediately by k consecutive up-steps) as a subpath,*
4. *partitions with n nonnegative parts bounded by the staircase partition $(n-1, n-2, \ldots, 1, 0)$ such that each positive number appears fewer than k times,*
5. *standard $2 \times n$ Young tableaux whose top row avoids contiguous labels of the form $i, j{+}1, j{+}2, \ldots, j{+}k$ for all $i < j$, and*
6. *permutations of $[n]$ avoiding 1-3-2 and $23\cdots(k+1)1$.*

Plane Trees

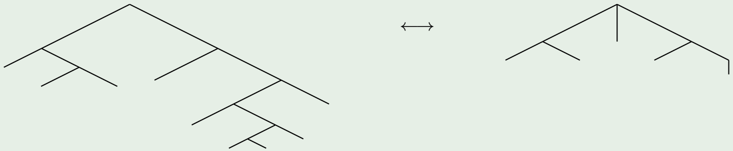
Definition

A *plane tree* is a rooted tree for which the children of each node are linearly ordered.

Fact

Binary trees with $n+1$ leaves correspond to plane trees with $n+1$ nodes via Knuth transform (left-child right-sibling representation of plane trees).

Example



Modular Catalan Objects

Theorem (Hein and H.)

The modular Catalan number $C_{n,k}$ enumerates the following objects:

1. *binary trees with $n+1$ leaves avoiding comb_k^1 as a subtree,*
2. *plane trees with $n+1$ nodes whose non-root nodes have degree less than k,*
3. *Dyck paths of length $2n$ avoiding DU^k (a down-step followed immediately by k consecutive up-steps) as a subpath,*
4. *partitions with n nonnegative parts bounded by the staircase partition $(n-1, n-2, \ldots, 1, 0)$ such that each positive number appears fewer than k times,*
5. *standard $2 \times n$ Young tableaux whose top row avoids contiguous labels of the form $i, j+1, j+2, \ldots, j+k$ for all $i < j$, and*
6. *permutations of $[n]$ avoiding 1-3-2 and $23\cdots(k+1)1$.*

Dyck Paths

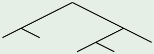
Definition

A *Dyck path of (semi-)length $2n$*, which is a diagonal lattice path from $(0,0)$ to $(2n,0)$ consisting of n up-steps $U=(1,1)$ and n down-steps $D=(1,-1)$ such that none of the path is below the x-axis.

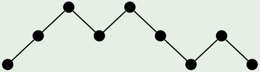
Fact

Binary trees with $n+1$ leaves correspond to Dyck paths of length $2n$.

Example



$\longleftrightarrow$



Modular Catalan Objects

Theorem (Hein and H.)

The modular Catalan number $C_{n,k}$ enumerates the following objects:

1. *binary trees with $n+1$ leaves avoiding comb_k^1 as a subtree,*
2. *plane trees with $n+1$ nodes whose non-root nodes have degree less than k,*
3. *Dyck paths of length $2n$ avoiding DU^k (a down-step followed immediately by k consecutive up-steps) as a subpath,*
4. *partitions with n nonnegative parts bounded by the staircase partition $(n-1, n-2, \ldots, 1, 0)$ such that each positive number appears fewer than k times,*
5. *standard $2 \times n$ Young tableaux whose top row avoids contiguous labels of the form $i, j+1, j+2, \ldots, j+k$ for all $i < j$, and*
6. *permutations of $[n]$ avoiding 1-3-2 and $23\cdots(k+1)1$.*

Partitions

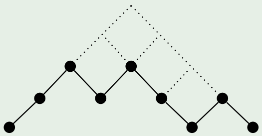
Definition

- A *partition* if a descreasing sequence of nonnegative integers.
- A partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ correspond to a *Young diagram* with λ_i boxes on its ith row.

Fact

Dyck paths of length $2n \leftrightarrow$ *partitions* $(\lambda_1, \ldots, \lambda_n)$ *with* $\lambda_i \leq n - i$.

Example



$$\longleftrightarrow \qquad \lambda = (3, 1, 0, 0)$$

Modular Catalan Objects

Theorem (Hein and H.)

The modular Catalan number $C_{n,k}$ enumerates the following objects:

1. *binary trees with $n+1$ leaves avoiding comb_k^1 as a subtree,*
2. *plane trees with $n+1$ nodes whose non-root nodes have degree less than k,*
3. *Dyck paths of length $2n$ avoiding DU^k (a down-step followed immediately by k consecutive up-steps) as a subpath,*
4. *partitions with n nonnegative parts bounded by the staircase partition $(n-1, n-2, \ldots, 1, 0)$ such that each positive number appears fewer than k times,*
5. *standard $2 \times n$ Young tableaux whose top row avoids contiguous labels of the form $i, j+1, j+2, \ldots, j+k$ for all $i < j$, and*
6. *permutations of $[n]$ avoiding 1-3-2 and $23\cdots(k+1)1$.*

Standard Tableaux

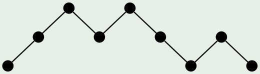
Definition

A *standard tableau of shape* λ is a filling of the Young diagram of λ with $1, 2, \ldots$ such that each row is increasing from left to right and each column is increasing from top to bottom.

Fact

Dyck paths of length $2n \leftrightarrow 2 \times n$ *standard tableaux.*

Example



$\longleftrightarrow$

1	2	4	7
3	5	6	8

Modular Catalan Objects

Theorem (Hein and H.)

The modular Catalan number $C_{n,k}$ enumerates the following objects:

1. *binary trees with $n+1$ leaves avoiding comb_k^1 as a subtree,*
2. *plane trees with $n+1$ nodes whose non-root nodes have degree less than k,*
3. *Dyck paths of length $2n$ avoiding DU^k (a down-step followed immediately by k consecutive up-steps) as a subpath,*
4. *partitions with n nonnegative parts bounded by the staircase partition $(n-1, n-2, \ldots, 1, 0)$ such that each positive number appears fewer than k times,*
5. *standard $2 \times n$ Young tableaux whose top row avoids contiguous labels of the form $i, j+1, j+2, \ldots, j+k$ for all $i < j$, and*
6. *permutations of $[n]$ avoiding 1-3-2 and $23\cdots(k+1)1$.*

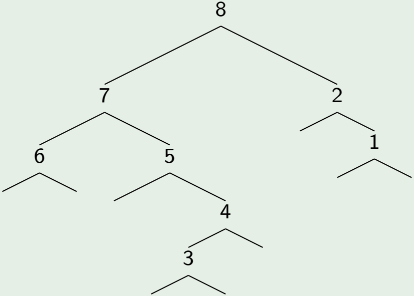
Permutations

Fact

Binary trees with $n+1$ *leaves* $\leftrightarrow$ *permutations of* $[n] := \{1, 2, \ldots, n\}$ *avoiding* 1-3-2.

Example

The picture below shows a binary tree with internal nodes labeled with [8]; reading these labels gives a permutation 67534821 avoiding 1-3-2.



Modular Catalan Objects

Theorem (Hein and H.)

The modular Catalan number $C_{n,k}$ enumerates the following objects:

1. *binary trees with $n+1$ leaves avoiding comb_k^1 as a subtree,*
2. *plane trees with $n{+}1$ nodes whose non-root nodes have degree less than k,*
3. *Dyck paths of length $2n$ avoiding DU^k (a down-step followed immediately by k consecutive up-steps) as a subpath,*
4. *partitions with n nonnegative parts bounded by the staircase partition $(n-1, n-2, \ldots, 1, 0)$ such that each positive number appears fewer than k times,*
5. *standard $2 \times n$ Young tableaux whose top row avoids contiguous labels of the form $i, j{+}1, j{+}2, \ldots, j{+}k$ for all $i < j$, and*
6. *permutations of $[n]$ avoiding 1-3-2 and $23\cdots(k+1)1$.*

For $n \geq 0$ and $k \geq 1$, the *generalized Motzkin number* $M_{n,k}$ enumerates

1. binary trees with $n+1$ leaves avoiding comb_k as a subtree,
2. plane trees with $n+1$ nodes, each having degree less than k [Takácz],
3. Dyck paths of length $2n$ avoiding U^k (k consecutive up-steps).
4. partitions with n parts bounded by $(n-1, n-2, \ldots, 1, 0)$ such that each number appears fewer than k times,
5. $2 \times n$ standard Young tableaux avoiding k consecutive numbers in the top row, and
6. permutations of $[n]$ avoiding 1-3-2 and $12 \cdots k$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	OEIS
$M_{n,1}$	1	0	0	0	0	0	0	0	0	0	0	0	0	A000007
$M_{n,2}$	1	1	1	1	1	1	1	1	1	1	1	1	1	A000012
$M_{n,3}$	1	1	2	4	9	21	51	127	323	835	2188	5798	15511	A001006
$M_{n,4}$	1	1	2	5	13	36	104	309	939	2905	9118	28964	92940	A036765
$M_{n,5}$	1	1	2	5	14	41	125	393	1265	4147	13798	46476	158170	A036766
$M_{n,6}$	1	1	2	5	14	42	131	421	1385	4642	15795	54418	189454	A036767
$M_{n,7}$	1	1	2	5	14	42	132	428	1421	4807	16510	57421	201824	A036768
$M_{n,8}$	1	1	2	5	14	42	132	429	1429	4852	16730	58422	206192	A036769
C_n	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	A000108

Modular Catalan Objects

Theorem (Hein and H.)

The modular Catalan number $C_{n,k}$ enumerates the following objects:

1. *binary trees with $n+1$ leaves avoiding comb_k^1 as a subtree,*
2. *plane trees with $n+1$ nodes whose non-root nodes have degree less than k,*
3. *Dyck paths of length $2n$ avoiding DU^k (a down-step followed immediately by k consecutive up-steps) as a subpath,*
4. *partitions with n nonnegative parts bounded by the staircase partition $(n-1, n-2, \ldots, 1, 0)$ such that each positive number appears fewer than k times,*
5. *standard $2 \times n$ Young tableaux whose top row avoids contiguous labels of the form $i, j+1, j+2, \ldots, j+k$ for all $i < j$, and*
6. *permutations of $[n]$ avoiding 1-3-2 and $23 \cdots (k+1)1$.*

Proposition (Hein and H.)

- *Recurrence:* $M_{n,k} = \sum_{0\le\ell<k} \sum_{n_1+\cdots+n_\ell=n-\ell} M_{n_1,k}\cdots M_{n_\ell,k}$.
- *Generating function:* $M_k(x) := \sum_{n\ge0} M_{n,k}$ *satisfies*

$$M_k(x) = 1 + xM_k(x) + x^2M_k(x)^2 + \cdots + x^{k-1}M_k(x)^{k-1}.$$

- *Closed formula (applying Lagrange inversion to the above equation):*

$$\begin{aligned} M_{n,k} &= \frac{1}{n+1} \sum_{\substack{|\lambda|=n \\ \lambda\subseteq(k-1)^{n+1}}} m_\lambda(\underbrace{1,\ldots,1}_{n+1}) \\ &= \frac{1}{n+1} \sum_{0\le j\le n/k} (-1)^j \binom{n+1}{j}\binom{2n-jk}{n}. \end{aligned}$$

Proposition (Hein and H.)

The generating function $C_k(x) := \sum_{n\geq 0} C_{n,k}$ *satisfies*

$$C_k(x) = \frac{1}{1 - xM_k(x)} = \sum_{\ell\geq 0} (xM_k(x))^\ell.$$

Theorem (Hein and H.)

- *For* $n, k \geq 1$ *we have*

$$\begin{aligned} C_{n,k} &= \sum_{\substack{\lambda\subseteq(k-1)^n \\ |\lambda|<n}} \frac{n-|\lambda|}{n} m_\lambda(\underbrace{1,\ldots,1}_{n}) \\ &= \sum_{0\leq j(n-1)/k} \frac{(-1)^j}{n}\binom{n}{j}\binom{2n-jk}{n+1}. \end{aligned}$$

- *We also have* $x(C_k(x)-1)^k - xC_k(x)^k + C_k(x)^{k-1} - C_k(x)^{k-2} = 0.$

Combinatorial Proofs

- Let $w = U^{i_0} D U^{i_1} D U^{i_2} \cdots D U^{i_n}$, with $i_0 > 0$, $i_1, \ldots, i_n \geq 0$, and $i_0 + i_1 + \cdots + i_n = n$.
- Rotation: $w^{*j} := U^{i_0} D U^{i_{j+1}} \cdots D U^{i_n} D U^{i_1} \cdots D U^{i_j}$.
- 

$w = U^2 DDUDDU$ $\quad w^{*1} = U^2 DUDDUD$ $\quad w^{*2} = U^2 DDUDDU$ $\quad w^{*3} = U^2 DUDDUD$

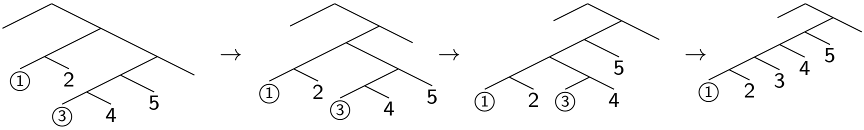
- $\#\{j \in \{0, 1, \ldots, n-1\} : w^{*j} \text{ is a Dyck path}\} = i_0$.
- This rotation implies the first formula for $C_{n,k}$.
- Coloring one copy of U among U^{i_0} by blue and assuming DU^k occurs at least j times, we get $\binom{n}{j}\binom{2n-jk}{n+1}$ many lattice paths.
- Applying the above rotation and using inclusion-exclusion we prove the second formula for $C_{n,k}$.

2-Modular Catalan Numbers

- The positive sum formula for $C_{n,k}$ becomes the following when $k = 2$:

$$2^{n-1} = \sum_{0 \le i \le n-1} \binom{n-1}{i}.$$

- The $C_{n,2} = 2^{n-1}$ binary trees with $n+1$ leaves avoiding comb_2^1 form a lattice under the *Tamari order*, which is isomorphic to the Boolean algebra of subsets of $[n]$ ordered by inclusion.
- The rank of a binary tree avoiding comb_2^1 equals the number of non-root internal nodes on its right border.
-



3-Modular Catalan Numbers

- The 3-modular Catalan numbers $\{C_{n,3}\}$ count many other objects:
  1. directed n-ominoes in standard position,
  2. n-digit base three numbers whose digits sum to n,
  3. permutations of $[n]$ avoiding 1-3-2 and 123-4,
  4. minimax elements in the affine Weyl group of the Lie algebra $\mathfrak{so}_{2n+1}$.
- Our positive sum formula for $C_{n,3}$ can be simplified to

$$C_{n,3} = \sum_{0\le i\le n-1} \binom{n-1}{i}\binom{i}{\lfloor i/2\rfloor}$$

which was obtained by Gouyou-Beauchamps and Viennot in their study of the objects in ① and by Panyushev in his study of objects ④.

Questions

- How is the simplified formula for $C_{n,3}$ related to the modular Catalan objects? Is it possible to generalize this formula to all $k \geq 1$?
- Let $\mathcal{T}_{n,k} := \{$binary trees with $n+1$ leaves avoiding $\mathrm{comb}_k^1\}$ be a subposet of the Tamari lattice. What can be said about this poset? How is this poset related to the $(n-1)$-dimensional associahedron?
- Other modular Catalan objects (noncrossing partitions, triangulations of convex polygons, etc.)?
- Other binary operations?
- We know $1 \leq C_{n,*} \leq C_n$, and $1 = C_{n,*}$ if and only if $*$ is associative. When does $C_{n,*} = C_n$ hold?

Thank you!