Non-Associativity of the Double Minus Operation

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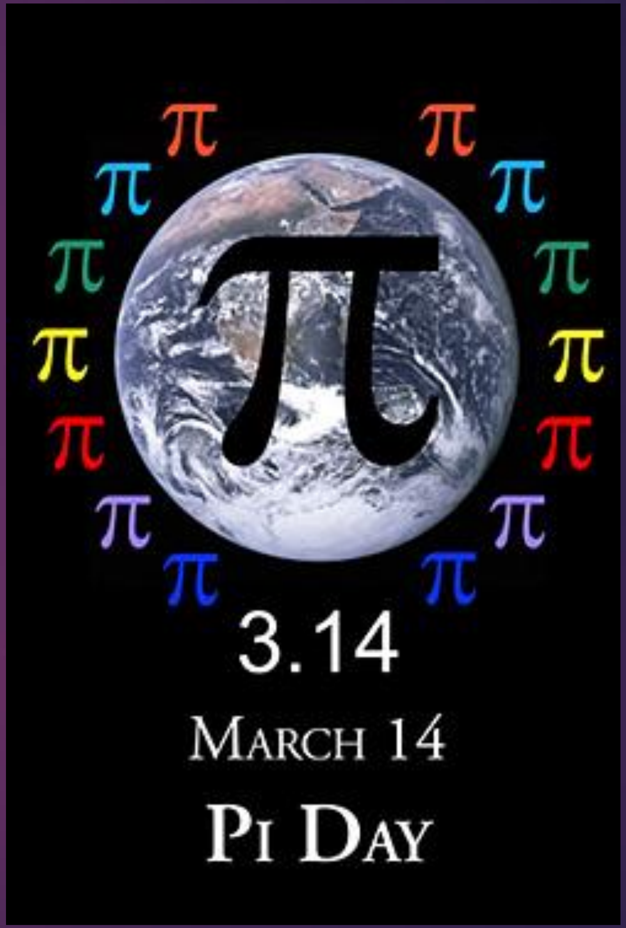
JOINT WORK WITH J. HUANG AND J. XU



Double Minus Operation

- Define binary operation: $a \ominus b = -a - b$
- Let $C_{\ominus,n,r}$ be the number of distinct results with exactly r plus signs from

$$x_0 \ominus x_1 \ominus \ldots \ominus x_n$$

- Let $C_{\ominus,n} = \sum_{0 \leq r \leq n+1} C_{\ominus,n,r}$ for $n \geq 0$ which is the total number of distinct results from the previous expression.
- $C_{\ominus,n}$ measures the non-associativity of the operation $\ominus$

Example (n = 3)

$-(-(-x_0 - x_1) - x_2) - x_3 = -x_0 - x_1 + x_2 - x_3$ **- - + -**

$-(-x_0 - x_1) - (-x_2 - x_3) = x_0 + x_1 + x_2 + x_3$ **+ + + +**

$-\left(-x_0 - (-x_1 - x_2)\right) - x_3 = x_0 - x_1 - x_2 - x_3$ **+ - - -**

$-x_0 - (-(-x_1 - x_2) - x_3) = -x_0 - x_1 - x_2 + x_3$ **- - - +**

$-x_0 - \left(-x_1 - (-x_2 - x_3)\right) = -x_0 + x_1 - x_2 - x_3$ **- + - -**

$$\left.\begin{aligned} C_{\theta,3,0} &= 0 \\ C_{\theta,3,1} &= 4 = \binom{4}{1} \\ C_{\theta,3,2} &= 0 \\ C_{\theta,3,3} &= 0 \\ C_{\theta,3,4} &= 1 = \binom{4}{4} \end{aligned}\right\} \qquad C_{\theta,3} = \binom{4}{1} + \binom{4}{4} = 5$$

Theorem (Huang, M., and Xu)

For $n \geq 1$ and $0 \leq r \leq n+1$:

$$C_{\Theta,n,r} = \begin{cases} \binom{n+1}{r} & \text{if } n+r \equiv 1 \pmod 3 \text{ and } n \neq 2r-2; \\ \binom{n+1}{r} - 1 & \text{if } n+r \equiv 1 \pmod 3 \text{ and } n = 2r-2; \\ 0 & \text{if } n+r \not\equiv 1 \pmod 3. \end{cases}$$

Theorem (Huang, M., and Xu)

For $n \geq 1$ we have

$$C_{\Theta,n} = \begin{cases} \frac{2^{n+1} - 1}{3}, & if\ n\ is\ odd \\ \frac{2^{n+1} - 2}{3}, & if\ n\ is\ even. \end{cases}$$

Proof by Example

- $\binom{6}{0} + \binom{6}{3} + \binom{6}{6} = ?$ $\binom{6}{1} + \binom{6}{4} = ?$ $\binom{6}{2} + \binom{6}{5} = ?$
- Using the binomial expansion:
  - $(1+x)^6 = \binom{6}{0} + \binom{6}{1}x^1 + \binom{6}{2}x^2 + \binom{6}{3}x^3 + \binom{6}{4}x^4 + \binom{6}{5}x^5 + \binom{6}{6}x^6$
  - $x = 1$: $\binom{6}{0} + \binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4} + \binom{6}{5} + \binom{6}{6}$
  - $x = -1$: $\binom{6}{0} - \binom{6}{1} + \binom{6}{2} - \binom{6}{3} + \binom{6}{4} - \binom{6}{5} + \binom{6}{6}$
  - Add together and divide by 2: $\binom{6}{0} + \binom{6}{2} + \binom{6}{4} + \binom{6}{6}$
    - Result based on $(\pm 1) = 1$ and $1 + (-1) = 0$ (Second Roots of Unity)
- Used Third Roots of Unity to calculate binomial coefficient totals
  - $1^3 = 1, \omega^3 = 1, (\omega^2)^3 = 1$ and $1 + \omega + \omega^2 = 0$

Examining Non-Associativity

- Related to Sequence A000975 in the OEIS (On-line Encyclopedia of Integer Sequences)
- This sequence is characterized by a recurrence relation.
- This sequence also gives binary operations with alternating ones and zeros.

$$A(1) = 1 = 1_2 = 1 \times 2^0$$

$$A(2) = 2 = 10_2 = 1 \times 2^1 + 0 \times 2^0$$

$$A(3) = 5 = 101_2 = 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$$

$$A(4) = 10 = 1010_2 = 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 0 \times 2^0$$

$$A(5) = 21 = 10101_2 = 1 \times 2^4 + 0 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$$

- This sequence occurs in other interesting places.

THANK YOU!