0-Hecke algebra actions on quotients of polynomial rings

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The Symmetric Group $\mathfrak{S}_n$

- The \textit{symmetric group} $\mathfrak{S}_n := \{\text{bijections on } \{1, \ldots, n\}\}$ is generated by the \textit{adjacent transpositions} $s_i = (i, i + 1)$, $1 \leq i \leq n - 1$, with quadratic relations $s_i^2 = 1$, $1 \leq i \leq n - 1$, and braid relations

$$\begin{cases}
  s_is_{i+1}s_i = s_{i+1}s_is_{i+1}, & 1 \leq i \leq n - 2, \\
  s_is_j = s_js_i, & |i - j| > 1.
\end{cases}$$

- More generally, a \textit{Coxeter group} has a similar presentation.

- The \textit{length} of any $w \in \mathfrak{S}_n$ is $\ell(w) := \min\{k : w = s_{i_1} \cdots s_{i_k}\}$, which coincides with $\text{inv}(w) := \{(i, j) : 1 \leq i < j \leq n, w(i) > w(j)\}$.

- For example, $w = 3241 \in \mathfrak{S}_4$ has $\ell(w) = \text{inv}(w) = 4$ and reduced repressions $w = s_2s_1s_2s_3 = s_1s_2s_1s_3 = s_1s_2s_3s_1$. 

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The Hecke Algebra $H_n(q)$

- The \textit{(Iwahori-)Hecke algebra} $H_n(q)$ is a deformation of the group algebra $\mathbb{F}\mathcal{S}_n$ of $\mathcal{S}_n$ over an arbitrary field $\mathbb{F}$.

- It is an $\mathbb{F}(q)$-algebra generated by $T_1, \ldots, T_{n-1}$ with relations

  \[
  \begin{cases}
  (T_i + 1)(T_i - q) = 0, & 1 \leq i \leq n - 1, \\
  T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, & 1 \leq i \leq n - 2, \\
  T_i T_j = T_j T_i, & |i - j| > 1.
  \end{cases}
  \]

- It has an $\mathbb{F}(q)$-basis $\{T_w : w \in \mathcal{S}_n\}$, where $T_w := T_{s_1} \cdots T_{s_k}$ if $w = s_1 \cdots s_k$ with $k$ minimum.

- It has significance in algebraic combinatorics, knot theory, quantum groups, representation theory of p-adic groups, etc.
The 0-Hecke algebra $H_n(0)$

- Set $q = 1$: $H_n(q) \rightarrow \mathbb{F}S_n$, $T_i \rightarrow s_i$, $T_w \rightarrow w$.
- Tits showed that $H_n(q) \cong \mathbb{C}S_n$ unless $q \in \{0, \text{roots of unity}\}$.
- Set $q = 0$: $H_n(q) \rightarrow H_n(0)$, $T_i \rightarrow \pi_i$, $T_w \rightarrow \pi_w$,
  \[
  \begin{cases}
  \pi_i^2 = -\pi_i, & 1 \leq i \leq n - 1, \\
  \pi_i \pi_{i+1} \pi_i = \pi_{i+1} \pi_i \pi_{i+1}, & 1 \leq i \leq n - 2, \\
  \pi_i \pi_j = \pi_j \pi_i, & |i - j| > 1.
  \end{cases}
  \]
- $H_n(0)$ has another generating set $\{\pi_i := \pi_i + 1\}$, with relations
  \[
  \begin{cases}
  \pi_i^2 = \pi_i, & 1 \leq i \leq n - 1, \\
  \pi_i \pi_{i+1} \pi_i = \pi_{i+1} \pi_i \pi_{i+1}, & 1 \leq i \leq n - 2, \\
  \pi_i \pi_j = \pi_j \pi_i, & |i - j| > 1.
  \end{cases}
  \]
- Sending $\pi_i$ to $-\pi_i$ gives an algebra automorphism.
Significance of the 0-Hecke algebra

- Using the automorphism $\pi_i \mapsto -\pi_i$ of $H_n(0)$, Stembridge (2007) gave a short derivation for the Möbius function of the Bruhat order of $\mathfrak{S}_n$ (or more generally, any Coxeter group).

- Norton (1979) studied the representation theory of $H_n(0)$ over an arbitrary field $\mathbb{F}$.


- Krob and Thibon (1997) discovered connections between $H_n(0)$-representations and certain generalizations of symmetric functions, which is similar to the classical Frobenius correspondence between $\mathfrak{S}_n$-representations and symmetric functions.
Analogies between $\mathcal{S}_n$ and $H_n(0)$

- $\mathbb{F}\mathcal{S}_n$ is the group algebra of the symmetric group $\mathcal{S}_n$ and $H_n(0)$ is the monoid algebra of the monoid $\{\pi_w : w \in W\}$.
- The defining representations of $\mathcal{S}_n$ and $H_n(0)$ are analogous:

$$
\begin{align*}
1 & \xleftarrow{s_1} 2 \xleftarrow{s_2} \cdots \xleftarrow{s_{n-1}} n \\
1 & \xrightarrow{\pi_1} 2 \xrightarrow{\pi_2} \cdots \xrightarrow{\pi_{n-1}} n
\end{align*}
$$

- $\mathcal{S}_n$ acts on $\mathbb{Z}^n$: $s_i$ swaps $a_i$ and $a_{i+1}$ in $a_1 \cdots a_n$.
- $H_n(0)$ acts on $\mathbb{Z}^n$ by the bubble-sorting operators: $\pi_i$ swaps $a_i$ and $a_{i+1}$ in $a_1 \cdots a_n$ if $a_i > a_{i+1}$, or fixes $a_1 \cdots a_n$ otherwise.
- Analogies between other representations of $\mathcal{S}_n$ and $H_n(0)$?
\textbf{Actions on polynomials}

- $\mathcal{S}_n$ acts on $\mathbb{F}[X] := \mathbb{F}[x_1, \ldots, x_n]$ by variable permutation.
- $H_n(0)$ also acts on $\mathbb{F}[X]$ via the \textit{Demazure operators}
  \[ \pi_i(f) := \partial_i(x_i f) = \frac{x_i f - s_i(x_i f)}{x_i - x_{i+1}}. \]

- The \textit{divided difference operator} $\partial_i$ is useful in Schubert calculus, a branch of algebraic geometry.
- $\pi_1(x_1^3 x_2 x_3 x_4^4) = (x_1^3 x_2 + x_1^2 x_2^2 + x_1 x_2^3) x_3 x_4^4$.
- $\pi_2(x_1^3 x_2 x_3 x_4^4) = x_1^3 x_2 x_3 x_4^4$.
- $\pi_3(x_1^3 x_2 x_3 x_4^4) = x_1^3 x_2 (-x_3^2 x_4^3 - x_3 x_4^2)$. 
The coinvariant algebra of $\mathfrak{S}_n$

- The **invariant ring** $\mathbb{F}[X]^{\mathfrak{S}_n} := \{ f \in \mathbb{F}[X] : wf = f, \forall w \in \mathfrak{S}_n \}$ consists of all symmetric functions in $x_1, \ldots, x_n$. It is a polynomial ring $\mathbb{F}[X]^{\mathfrak{S}_n} = \mathbb{F}[e_1, \ldots, e_n]$ in the **elementary symmetric functions**

$$e_k := \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} \cdots x_{i_k}, \quad k = 1, \ldots, n.$$  

$n = 3$: $e_1 = x_1 + x_2 + x_3$, $e_2 = x_1x_2 + x_1x_3 + x_2x_3$, $e_3 = x_1x_2x_3$

- If $f \in \mathbb{F}[X]^{\mathfrak{S}_n}$ and $g \in \mathbb{F}[X]$, then $s_i(fg) = fs_i(g)$.
- Thus $\mathbb{F}[X]/(e_1, \ldots, e_n)$ becomes a graded $\mathfrak{S}_n$-module.

**Theorem (Chevalley–Shephard–Tod 1955, indirect proof)**

*The coinvariant algebra $\mathbb{F}[X]/(e_1, \ldots, e_n)$ is isomorphic to the regular representation $\mathbb{F}\mathfrak{S}_n$ of $\mathfrak{S}_n$, if $\mathbb{F}$ is a field of characteristic 0.*
The coinvariant algebra of $H_n(0)$

- The $H_n(0)$-invariants are also the symmetric functions: $\pi_i f = f$ if and only if $s_i f = f$ for all $i$.
- If $f \in \mathbb{F}[X]^{S_n}$ and $g \in \mathbb{F}[X]$, then $\pi_i (fg) = f \pi_i (g)$.
- Thus $\mathbb{F}[X]/(e_1, \ldots, e_n)$ becomes a graded $H_n(0)$-module.

**Theorem (H. 2014)**

The coinvariant algebra $\mathbb{F}[X]/(e_1, \ldots, e_n)$ is isomorphic to the regular representation of $H_n(0)$.

**Remark**

Our proof is constructive, using the *descent basis* of the coinvariant algebra given by Garsia and Stanton (1984).
$H_3(0) \cong \mathbb{F}[x_1, x_2, x_3]/(e_1, e_2, e_3)$
Every $\mathcal{S}_n$-module is a direct sum of simple modules.

A **partition** of $n$ is a decreasing sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$ of positive integers whose sum is $n$; this is denoted by $\lambda \vdash n$.

The simple $\mathcal{S}_n$-modules $S^\lambda$ are indexed by partitions $\lambda \vdash n$.

The **Schur function** $s_\lambda$ is the sum of $x_\tau$ for all semistandard tableaux $\tau$ of shape $\lambda$. For example,

$$s_{21} = x_{\begin{array}{c}1 \\ 2 \end{array}} + x_{\begin{array}{c}1 \\ 2 \end{array}} + \cdots = x_1^2 x_2 + x_1 x_2^2 + \cdots.$$

Symmetric functions form a graded Hopf algebra with a self-dual basis $\{s_\lambda\}$.

The **Frobenius characteristic map** $S^\lambda \mapsto s_\lambda$ is an isomorphism from $\mathcal{S}_n$-representations to $\text{Sym}$.
A composition of $n$, denoted by $\alpha \models n$, is a sequence $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ of positive integers whose sum is $n$.

Norton (1979) showed that $H_n(0) = \bigoplus_{\alpha \models n} P_\alpha$, so every projective indecomposable $H_n(0)$-module is isomorphic to $P_\alpha$ for some $\alpha \models n$.

Furthermore, every simple $H_n(0)$-module is isomorphic to some $C_\alpha := \text{top}(P_\alpha) = P_\alpha/\text{rad} P_\alpha$, which is 1-dimensional.

Generalizing $\text{Sym}$ are two graded Hopf algebras $\text{QSym}$ (quasisymmetric functions) and $\text{NSym}$ (noncommutative symmetric functions) with dual bases $\{F_\alpha\}$ and $\{s_\alpha\}$. We have $\text{NSym} \rightarrow \text{Sym} \hookrightarrow \text{QSym}$.

Krob and Thibon (1997): by $P_\alpha \leftrightarrow s_\alpha$ and $C_\alpha \leftrightarrow F_\alpha$ one has

\{ $H_n(0)$-modules $\} \leftrightarrow \text{QSym}$ (up to composition factors),
\{ projective $H_n(0)$-modules $\} \leftrightarrow \text{NSym}$.
$H_3(0) \cong \mathbb{F}[x_1, x_2, x_3]/(e_1, e_2, e_3)$
\( \alpha = (1, 2, 1) \)
A generalization of the coinvariant algebra

- Let $n \geq k \geq 1$ be two integers. Define a homogeneous ideal
  \[ I_{n,k} := \langle x_1^k, x_2^k, \ldots, x_n^k, e_n, e_{n-1}, \ldots, e_{n-k+1} \rangle. \]

- The span of $x_1^k, x_2^k, \ldots, x_n^k$ is isomorphic to the defining representation of $\mathfrak{S}_n$.

- The quotient $R_{n,k} := \mathbb{C}[X]/I_{n,k}$ is a graded $\mathfrak{S}_n$-module.

- The coinvariant algebra $\mathbb{C}[X]/(e_1, \ldots, e_n)$ is $R_{n,n}$. 

\[ 1 \leftarrow s_1 \rightarrow 2 \leftarrow s_2 \rightarrow \cdots \leftarrow s_{n-1} \rightarrow n \]

\[ x_1^k \leftarrow s_1 \rightarrow x_2^k \leftarrow s_2 \rightarrow \cdots \leftarrow s_{n-1} \rightarrow x_n^k \]
The $\mathfrak{S}_n$-module structure of $R_{n,k}$

- Let $\mathcal{OP}_{n,k}$ be the set of all $k$-block partitions of the set $[n]$. For example, $(35|126|4) \in \mathcal{OP}_{6,3}$.
- We have $|\mathcal{OP}_{n,k}| = k! \cdot \text{Stir}(n, k)$, where $\text{Stir}(n, k)$ is the \textit{(signless) Stirling number of the second kind}.
- Let $\text{SYT}(n)$ be the set of \textit{standard Young tableaux} of size $n$.

**Theorem (Haglund–Rhoades–Shimozono 2017+)**

As an ungraded $\mathfrak{S}_n$-module, $R_{n,k}$ is isomorphic to $\mathbb{C}[\mathcal{OP}_{n,k}]$. Moreover, the graded Frobenius characteristic of $R_{n,k}$ is

$$\sum_{\tau \in \text{SYT}(n)} q^{\text{maj}(\tau)} \left( d - \text{des}(\tau) - 1 \right)^{n-k} \text{s}_{\text{shape}(\tau)}.$$
A 0-Hecke analogue

- Define $J_{n,k}$ to be the ideal of $\mathbb{F}[X]$ generated by elementary symmetric functions $e_n, e_{n-1}, \ldots, e_{n-k+1}$ and complete homogeneous symmetric functions $h_k(x_1), h_k(x_1, x_2), \ldots, h_k(x_1, x_2, \ldots, x_n)$.

- The span of $h_k(x_1), h_k(x_1, x_2), \ldots, h_k(x_1, x_2, \ldots, x_n)$ is isomorphic to the defining representation of $H_n(0)$.

  \[
  1 \xrightarrow{\pi_1} 2 \xrightarrow{\pi_2} \cdots \xrightarrow{\pi_{n-1}} n
  \]

  \[
  h_k(x_1) \xrightarrow{\pi_1} h_k(x_1, x_2) \xrightarrow{\pi_2} \cdots \xrightarrow{\pi_{n-1}} h_k(x_1, \ldots, x_n)
  \]

- The quotient $S_{n,k} := \mathbb{F}[X]/J_{n,k}$ is a graded $H_n(0)$-module.

**Theorem (H.–Rhoades 2017+)**

As an ungraded $H_n(0)$-module, $S_{n,k}$ is isomorphic to $\mathbb{F}[\mathcal{OP}_{n,k}]$. 
A decomposition of $\mathbb{F}[\mathcal{OP}_{4,2}]$
A decomposition of $S_{4,2}$

\[
P_4 \oplus P_{13}
\]

\[
\begin{align*}
\pi_1 &= 0 \\
\pi_2 &= 0 \\
\pi_3 &= 0
\end{align*}
\]
Theorem (H.–Rhoades 2017+)

The graded $H_n(0)$-module $S_{n,k}$ corresponds

$$\sum_{\alpha \vdash n} t^{\text{maj}(\alpha)} \left[ \frac{n - \ell(\alpha)}{k - \ell(\alpha)} \right] s_\alpha \quad \text{inside NSym}$$

and its graded quasisymmetric characteristic coincides with the graded Frobenius characteristics of the $S_n$-module $R_{n,k}$.

Remark

This result connects to the Delta Conjecture of Haglund, Remmel, and Wilson (2016) in the theory of Macdonald polynomials.
More quotients of the polynomial ring

Theorem (DeConcini, Garsia, Procesi, Hotta, Springer, Tanisaki)

- For any $\mu \vdash n$, $\mathbb{C}[X]$ has a homogeneous $S_n$-stable ideal $J_\mu$ generated by certain elementary symmetric functions in partial variable sets.
- $R_\mu = \mathbb{C}[X]/J_\mu$ is isomorphic to the cohomology ring of the Springer fiber indexed by $\mu$.
- The graded Frobenius characteristic of $R_\mu = \mathbb{C}[X]/J_\mu$ is the modified Hall-Littlewood symmetric function

$$\tilde{H}_\mu(x; t) = \sum_\lambda t^{n(\mu)}K_{\lambda\mu}(t^{-1})s_\lambda$$

where $n(\mu) = \mu_2 + 2\mu_3 + 3\mu_4 + \cdots$ and $K_{\lambda\mu}(t)$ is the Kostka-Foulkes polynomial.
Theorem (H. 2014)

- The ideal $J_\mu$ is $H_n(0)$-stable if and only if $\mu = (1^k, n - k)$ is a hook. Assume $\mu$ is a hook below.
- Then $R_\mu = \mathbb{C}[X]/J_\mu$ becomes a projective $H_n(0)$-module.
- Its graded noncommutative characteristic is

$$\text{ch}_t(\mathbb{C}[X]/J_\mu) = \sum_{\alpha \text{ refined by } \mu} t^{\text{maj}(\alpha)} s_\alpha = \tilde{H}_\mu(x; t).$$

- Its graded quasisymmetric characteristic is

$$\text{Ch}_t(\mathbb{C}[X]/J_\mu) = \sum_{\alpha \text{ refined by } \mu} t^{\text{maj}(\alpha)} s_\alpha = \tilde{H}_\mu(x; t).$$
We introduced $H_n(0)$-actions on certain quotients of the Stanley-Reisner ring of the Boolean algebra [H. 2015]. This gives multigraded $H_n(0)$-modules which correspond to

- noncommutative analogues of $\tilde{H}_\mu(x; t)$ introduced by Bergeron–Zabrocki (2005) and Lascoux–Novelli–Thibon (2013),
- quasisymmetric generating function of the joint distribution of five permutation statistics studied by Garsia and Gessel (1979).

We studied the Stanley-Reisner ring of the Coxeter complex of any finite Coxeter group.

We are currently investigating a two-parameter family of quotients of the Stanley-Reisner ring (with Brendon Rhoades and Daniël Kroes).

Is there a nice $H_n(0)$-action on the Stanley-Reisner ring of the Tits building of a finite general linear group?
Thank you!