Domination ratio of infinite circulant graphs

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Domination in digraphs

- Let $\Gamma = (V, E)$ be a *digraph*, where V is a set and $E \subseteq V \times V$.
- Call $v \in V$ a *vertex* and $(u, v) \in E$ a *(directed) edge* from u to v.
- View Γ as undirected if $(u, v) \in E \Leftrightarrow (v, u) \in E$ for all $u, v \in V$.
- Given $u, v \in V$, we say u *dominates* v if $u = v$ or $(u, v) \in E$.
- A set $D \subseteq V$ is called a *dominating set* of the digraph Γ if every vertex $v \in V$ is dominated by some $u \in D$.
- The concept of domination has wide applications in real world, such as resource allocation.
- The *domination number* $\gamma(\Gamma)$ of a finite digraph Γ is the smallest cardinality of a dominating set of Γ.
- It is a well-known NP-complete problem to determine $\gamma(\Gamma)$.

Domination in finite Cayley graphs

- Given a group G and a set $S \subseteq G$, the *Cayley graph* $\Gamma(G, S) = (V, E)$ is a digraph with vertex set $V = G$ and edge set $E = \{(g, gs) : g \in G,\ s \in S\}$.
- If $\Gamma = \Gamma(G, S)$ is finite with $n = |G|$ vertices, each having $d = |S|$ outgoing edges, then
$$\gamma(\Gamma) \geq n/(1+d).$$
The equality holds iff Γ has an efficient dominating set.
- A dominating set D of Γ is *efficient* if every vertex of Γ is dominated by exactly one vertex in D.
- Work on efficient dominating sets in finite Cayley graphs: Dejter–Serra 2003, Chelvam–Mutharasu 2013, etc.

Domination in circulant graphs

- A *circulant (di)graph* is a finite Cayley graph $\Gamma(\mathbb{Z}_n, S)$, where $\mathbb{Z}_n$ is the finite cyclic group of integers modulo n and $S \subseteq \mathbb{Z}_n$.
- If $-S := \{-s : s \in S\}$ coincides with S then $\Gamma(\mathbb{Z}, S)$ can be viewed as an undirected graph.
- Properties: symmetry, fault-tolerance, routing capabilities.
- Applications: telecommunication networks, VLSI design, and distribute computation.
- Domination in circulant graphs: Obradović–Peters–Ružić 2007, Rad 2009, Kumar–MacGillivray 2013.
- Let $\gamma(\mathbb{Z}_n, S)$ be the domination number of $\Gamma(\mathbb{Z}_n, S)$.

Some explicit results

Proposition (H.–Xu 2008)

- *If* $1 < s < \lceil n/2 \rceil$ *then* $\lceil n/5 \rceil \le \gamma(\mathbb{Z}_n, \{\pm 1, \pm s\}) \le \lceil n/3 \rceil$ *and* $\Gamma(\mathbb{Z}_n, \{\pm 1, \pm s\})$ *has an efficient dominating set if and only if* $5 \mid n$ *and* $s \equiv \pm 2 \pmod 5$.
- *If* $1 < s < n$ *then* $\lceil n/3 \rceil \le \gamma(\mathbb{Z}_n, \{1, s\}) \le \lceil n/2 \rceil$ *and* $\Gamma(\mathbb{Z}_n, \{1, s\})$ *has an efficient dominating set if and only if* $3 \mid n$ *and* $s \equiv 2 \pmod 3$.
- *If* $1 \le s \le n-1$ *then* $\gamma(\mathbb{Z}_n, \{1, 2, \ldots, s\}) = \lceil n/(s+1) \rceil$.

Proposition (Rad 2009)

- *If* $n \not\equiv 4 \pmod 5$ *then* $\gamma(\mathbb{Z}_n, \{\pm 1, \pm 3\}) = \lceil n/5 \rceil$.
- *If* $n \equiv 4 \pmod 5$ *then* $\gamma(\mathbb{Z}_n, \{\pm 1, \pm 3\}) = \lceil n/5 \rceil + 1$.

Infinite circulant graphs

- An *infinite circulant (di)graph* is a Cayley graph $\Gamma(\mathbb{Z}, S)$ where $\mathbb{Z}$ is the infinite additive cyclic group of all integers and $S \subseteq \mathbb{Z}$.
- When $S = -S$ the undirected graph $\Gamma(\mathbb{Z}, S)$ is known as a *distance graph*, whose chromatic number has been extensively studied by many people, such as Erdös and Zhu.
- Infinite circulant graphs are natural extensions of finite circulant graphs and may shed light on their asymptotic behaviors.
- We may assume $0 \notin S$, since removing an edge from a vertex v to itself (i.e., a loop) has no effect on domination.
- If S is finite then a dominating set of $\Gamma(\mathbb{Z}, S)$ must be infinite since every vertex dominates at most $|S|$ many other vertices.

Domination ratio

- To measure how large a possibly infinite subset U of $\mathbb{Z}$ is, we define the *(lower) density* of U in $\mathbb{Z}$ as the limit inferior below:

$$\delta(U) := \liminf_{n\to\infty} \frac{|U \cap [-n, n]|}{2n+1}. \quad (1)$$

- For example, if U is finite then $\delta(U) = 0$ and $\delta(\mathbb{Z} \setminus U) = 1$. If $U = 2\mathbb{Z}$ then $\delta(U) = 1/2$. In general, one has $0 \le \delta(U) \le 1$.
- Define the *domination ratio* $\overline{\gamma}(\mathbb{Z}, S)$ of the graph $\Gamma(\mathbb{Z}, S)$ to be the infimum of $\delta(D)$ over all dominating sets D of $\Gamma(\mathbb{Z}, S)$.
- Carraher, Galvin, Hartke, Radcliffe, and Stolee (2016) used upper density to study independent sets in an undirected infinite circulant graph $\Gamma(\mathbb{Z}, \pm S)$.

Periodic sets

Example

The set $(5\mathbb{Z}+1)\cup(5\mathbb{Z}+4)$ has period 5 and density $2/5$.

Proposition (Carraher and H. 2017+)

- *Let S be a finite subset of $\mathbb{Z}\setminus\{0\}$. Then the domination ratio of $\Gamma(\mathbb{Z},S)$ is achieved by some periodic dominating set D.*
- *If p is the period of D then $\overline{\gamma}(\mathbb{Z},S)=\delta(D)=|D\cap[1,p]|/p$.*
- *The finite circulant graph $\Gamma(\mathbb{Z}_p,S_p)$ has a minimum dominating set $D\cap[1,p]$ and its domination number is*

$$\gamma(\mathbb{Z}_p,S_p)=|D\cap[1,p]|=\overline{\gamma}(\mathbb{Z},S)p$$

where S_p to be the set of all the least positive residues of elements in S modulo p.

Basic results on the domination ratio

Theorem (Carraher and H. 2017+)

- *If* $S \subseteq S' \subseteq \mathbb{Z} \setminus \{0\}$ *then* $\overline{\gamma}(\mathbb{Z}, S') \leq \overline{\gamma}(\mathbb{Z}, S) = \overline{\gamma}(\mathbb{Z}, -S)$.
- *If* $|S| \leq 1$ *then* $\overline{\gamma}(\mathbb{Z}, S) = 1/(1+|S|)$. *If* $2 \leq |S| < \infty$ *then* $1/(|S|+1) \leq \overline{\gamma}(\mathbb{Z}, S) \leq 1/2$.
- *If* S *is finite and there exists an efficient dominating set of* $\Gamma(\mathbb{Z}, S)$ *then* $\overline{\gamma}(\mathbb{Z}, S) = 1/(|S|+1)$.
- *If* $S = \{i_1(s+1)+1, i_2(s+1)+2, \ldots, i_s(s+1)+s\}$ *with* $i_1, \ldots, i_s \in \mathbb{Z}$ *then* $\overline{\gamma}(\mathbb{Z}, S) = 1/(s+1)$.
- *If* d *divides all elements of* S *then* $\overline{\gamma}(\mathbb{Z}, S/d) = \overline{\gamma}(\mathbb{Z}, S)$, *where* $S/d := \{s/d : d \in S\}$.

Corollary (Carraher and H. 2017+)

It follows that $\overline{\gamma}(\mathbb{Z}, \{1, 3k+2\}) = 1/3$ *for all* $k \in \mathbb{Z}$.

Block structure

Definition

- A dominating set $D = \{x_i : i \in \mathbb{Z}\}$ of $\Gamma(\mathbb{Z}, S)$ decomposes $\mathbb{Z}$ into a disjoint union of *blocks* $B_i = \{x_i, x_i + 1, \ldots, x_{i+1} - 1\}$ for all $i \in \mathbb{Z}$.
- The *block structure* of D is the infinite sequence $(|B_i| : i \in \mathbb{Z})$ of block sizes, which determines D up to a translation.

Proposition (Carraher and H. 2017+)

- *If $s = 3k + 1$ or $s = -3k$ then $\Gamma(\mathbb{Z}, \{1, s\})$ has a dominating set with block structure $(3^k\ 2)^\infty$. Hence*

$$\overline{\gamma}(\mathbb{Z}, \{1, 3k + 1\}) = \overline{\gamma}(\mathbb{Z}, \{1, -3k\}) \leq (k + 1)/(3k + 2).$$

- *If $s = 3k$ or $s = -3k + 1$ then $\Gamma(\mathbb{Z}, \{1, s\})$ has a dominating set with block structure $(3^{k-1}\ 4\ 3^{k-1}\ 1)^\infty$. Hence*

$$\overline{\gamma}(\mathbb{Z}, \{1, 3k\}) \leq \overline{\gamma}(\mathbb{Z}, \{1, -3k + 1\}) = 2k/(6k - 1).$$

Main result

Theorem (Carraher and H. 2017+)

Let k be a positive integer. Then we have

$$\overline{\gamma}(\mathbb{Z},\{1,3k+1\}) = \overline{\gamma}(\mathbb{Z},\{1,-3k\}) = (k+1)/(3k+2) \quad \textit{and}$$

$$\overline{\gamma}(\mathbb{Z},\{1,3k\}) = \overline{\gamma}(\mathbb{Z},\{1,-3k+1\}) = 2k/(6k-1).$$

Sketch of Proof.

- Proof 1: Group blocks into clusters according to domination. Then study the ratio of the union of all clusters contained in $[-n,n]$.
- Proof 2: Use a discharging method similar to previous work on independence ratio.
- Proof 1 does not rely on the fact that the domination ratio of $\Gamma(\mathbb{Z},S)$ is achieved by a periodic dominating set, whereas Proof 2 does. □

Corollaries

Corollary

Let $S := \{s, t\} \subseteq \mathbb{Z} \setminus \{0\}$ *with* $s \mid t$.

- *If* $t/s = 3k + 1$ *or* $t/s = -3k$ *for some positive integer* k *then* $\overline{\gamma}(\mathbb{Z}, S) = (k + 1)/(3k + 2)$.
- *If* $t/s = 3k$ *or* $t/s = -3k + 1$ *for some positive integer* k *then* $\overline{\gamma}(\mathbb{Z}, S) = 2k/(6k - 1)$.
- *Moreover,* $\Gamma(\mathbb{Z}, S)$ *has an efficient dominating set if and only if* $t/s \equiv 2 \pmod 3$.

Corollary

If $k > 0$ *is an integer then* $\gamma(\mathbb{Z}_{3k+2}, \{\pm 1\}) = \gamma(\mathbb{Z}_{3k+2}, \{1, 2\}) = k + 1$ *and* $\gamma(\mathbb{Z}_{6k-1}, \{1, 3k\}) = 2k$.

Questions

- Is there any intuitive explanation for the equalities below?

$$\gamma(\mathbb{Z},\{1,3k\}) = \gamma(\mathbb{Z},\{1,-3k+1\})$$
$$\gamma(\mathbb{Z},\{1,3k+1\}) = \gamma(\mathbb{Z},\{1,-3k\})$$

- Study $\overline{\gamma}(\mathbb{Z},S)$ if $S=\{s,t\}\subseteq \mathbb{Z}\setminus\{0\}$ with $s\nmid t$ or if $|S|>2$.
- How about other graphs, such as products of paths or cycles?
- What if $S\subseteq\mathbb{Z}\setminus\{0\}$ is defined in some kind of random way?
- Replace domination with total domination, independent domination, etc.

Thank you!