Integer Tillings and Domination Ratio

Jia Huang

University of Nebraska at Kearney
E-mail address: huangj2@unk.edu

June 2019

Tiling the Integers without overlaps

- A finite set $S \subseteq \mathbb{Z}$ *tiles the integers* if there exists a set $D \subseteq \mathbb{Z}$ such that $\mathbb{Z} = S \oplus D := \bigsqcup_{d \in D}(S + d)$.

Tiling the Integers without overlaps

- A finite set $S \subseteq \mathbb{Z}$ *tiles the integers* if there exists a set $D \subseteq \mathbb{Z}$ such that $\mathbb{Z} = S \oplus D := \bigsqcup_{d \in D}(S + d)$.
- Example: $S = \{0, 1, 2\}$ tiles the integers, but $S = \{0, 1, 3\}$ does not.

Tiling the Integers without overlaps

- A finite set $S \subseteq \mathbb{Z}$ *tiles the integers* if there exists a set $D \subseteq \mathbb{Z}$ such that $\mathbb{Z} = S \oplus D := \bigsqcup_{d \in D}(S + d)$.
- Example: $S = \{0, 1, 2\}$ tiles the integers, but $S = \{0, 1, 3\}$ does not.
- Newman (1977) determined whether a given set S with cardinality $|S| = p^a$ for some prime p can tile the integers.

Tiling the Integers without overlaps

- A finite set $S \subseteq \mathbb{Z}$ *tiles the integers* if there exists a set $D \subseteq \mathbb{Z}$ such that $\mathbb{Z} = S \oplus D := \bigsqcup_{d \in D}(S + d)$.
- Example: $S = \{0, 1, 2\}$ tiles the integers, but $S = \{0, 1, 3\}$ does not.
- Newman (1977) determined whether a given set S with cardinality $|S| = p^a$ for some prime p can tile the integers.
- Coven and Meyerowitz (1999) generalized the result of Newman to the case $|S| = p^a q^b$ for two primes p, q.

Tiling the Integers without overlaps

- A finite set $S \subseteq \mathbb{Z}$ *tiles the integers* if there exists a set $D \subseteq \mathbb{Z}$ such that $\mathbb{Z} = S \oplus D := \bigsqcup_{d \in D}(S + d)$.
- Example: $S = \{0, 1, 2\}$ tiles the integers, but $S = \{0, 1, 3\}$ does not.
- Newman (1977) determined whether a given set S with cardinality $|S| = p^a$ for some prime p can tile the integers.
- Coven and Meyerowitz (1999) generalized the result of Newman to the case $|S| = p^a q^b$ for two primes p, q.
- Example: $S = \{1, 4, 8, 13\}$ tiles the integers, but $S = \{1, 4, 8, 15\}$ does not.

Tiling the Integers without overlaps

- A finite set $S \subseteq \mathbb{Z}$ *tiles the integers* if there exists a set $D \subseteq \mathbb{Z}$ such that $\mathbb{Z} = S \oplus D := \bigsqcup_{d \in D}(S + d)$.
- Example: $S = \{0, 1, 2\}$ tiles the integers, but $S = \{0, 1, 3\}$ does not.
- Newman (1977) determined whether a given set S with cardinality $|S| = p^a$ for some prime p can tile the integers.
- Coven and Meyerowitz (1999) generalized the result of Newman to the case $|S| = p^a q^b$ for two primes p, q.
- Example: $S = \{1, 4, 8, 13\}$ tiles the integers, but $S = \{1, 4, 8, 15\}$ does not.
- Observation: We may assume $0 \in S$, without loss of generality.

Tiling the integers with overlaps

- If $\mathbb{Z} = S \oplus D$, the *density* of D must be $\delta(D) = 1/|S|$, where

$$\delta(D) := \liminf_{n\to\infty} \frac{|D \cap [-n, n]|}{2n+1}.$$

Tiling the integers with overlaps

- If $\mathbb{Z} = S \oplus D$, the *density* of D must be $\delta(D) = 1/|S|$, where

$$\delta(D) := \liminf_{n\to\infty} \frac{|D \cap [-n, n]|}{2n+1}.$$

- Example: Tiling $\mathbb{Z}$ with translates of $S = \{0, 1, 2\}$ has a density $1/3$.

Tiling the integers with overlaps

- If $\mathbb{Z} = S \oplus D$, the *density* of D must be $\delta(D) = 1/|S|$, where
$$\delta(D) := \liminf_{n\to\infty} \frac{|D \cap [-n, n]|}{2n+1}.$$
- Example: Tiling $\mathbb{Z}$ with translates of $S = \{0, 1, 2\}$ has a density $1/3$.
- If S cannot tile $\mathbb{Z}$, we can still look for a set $D \subseteq \mathbb{Z}$ with minimum density $\delta(D)$ such that $\mathbb{Z} = \bigcup_{d\in D}(S + d)$.

Tiling the integers with overlaps

- If $\mathbb{Z} = S \oplus D$, the *density* of D must be $\delta(D) = 1/|S|$, where

$$\delta(D) := \liminf_{n\to\infty} \frac{|D \cap [-n, n]|}{2n+1}.$$

- Example: Tiling $\mathbb{Z}$ with translates of $S = \{0, 1, 2\}$ has a density $1/3$.
- If S cannot tile $\mathbb{Z}$, we can still look for a set $D \subseteq \mathbb{Z}$ with minimum density $\delta(D)$ such that $\mathbb{Z} = \bigcup_{d\in D}(S + d)$.
- Such a set D is actually a minimum dominating set of the *integer distance graph* $\Gamma(\mathbb{Z}, S)$, which is a Cayley graph with vertex set $\mathbb{Z}$ and edge set $\{(n, n + s) : n \in \mathbb{Z}, s \in S\}$.

Tiling the integers with overlaps

- If $\mathbb{Z} = S \oplus D$, the *density* of D must be $\delta(D) = 1/|S|$, where
$$\delta(D) := \liminf_{n\to\infty} \frac{|D \cap [-n, n]|}{2n+1}.$$
- Example: Tiling $\mathbb{Z}$ with translates of $S = \{0, 1, 2\}$ has a density $1/3$.
- If S cannot tile $\mathbb{Z}$, we can still look for a set $D \subseteq \mathbb{Z}$ with minimum density $\delta(D)$ such that $\mathbb{Z} = \bigcup_{d\in D}(S + d)$.
- Such a set D is actually a minimum dominating set of the *integer distance graph* $\Gamma(\mathbb{Z}, S)$, which is a Cayley graph with vertex set $\mathbb{Z}$ and edge set $\{(n, n+s) : n \in \mathbb{Z}, s \in S\}$.
- The independence ratio of an integer distance graph is closely related to its chromatic number and has been extensively studied.

Domination Ratio of an integer distance graph

- Say a vertex *dominates* itself and its (out-)neighbors.

Domination Ratio of an integer distance graph

- Say a vertex *dominates* itself and its (out-)neighbors.
- A *dominating set* D of a graph is a set of vertices such that every vertex in the graph is dominated by some element of D.

Domination Ratio of an integer distance graph

- Say a vertex *dominates* itself and its (out-)neighbors.
- A *dominating set* D of a graph is a set of vertices such that every vertex in the graph is dominated by some element of D.
- Finding minimum dominating sets is an NP-complete problem with many applications (e.g., resource allocation).

Domination Ratio of an integer distance graph

- Say a vertex *dominates* itself and its (out-)neighbors.
- A *dominating set* D of a graph is a set of vertices such that every vertex in the graph is dominated by some element of D.
- Finding minimum dominating sets is an NP-complete problem with many applications (e.g., resource allocation).
- The *domination ratio* $\bar{\gamma}(\mathbb{Z}, S)$ of the integer distance graph $\Gamma(\mathbb{Z}, S)$ is the infimum of $\delta(D)$ over all dominating sets D of $\Gamma(\mathbb{Z}, S)$.

Domination Ratio of an integer distance graph

- Say a vertex *dominates* itself and its (out-)neighbors.
- A *dominating set* D of a graph is a set of vertices such that every vertex in the graph is dominated by some element of D.
- Finding minimum dominating sets is an NP-complete problem with many applications (e.g., resource allocation).
- The *domination ratio* $\bar{\gamma}(\mathbb{Z}, S)$ of the integer distance graph $\Gamma(\mathbb{Z}, S)$ is the infimum of $\delta(D)$ over all dominating sets D of $\Gamma(\mathbb{Z}, S)$.
- We have $\mathbb{Z} = S \oplus D$ if and only if every vertex in $\Gamma(\mathbb{Z}, S)$ is dominated by exactly one element of D, i.e., D is an *efficient dominating set*.

Domination Ratio of an integer distance graph

- Say a vertex *dominates* itself and its (out-)neighbors.
- A *dominating set* D of a graph is a set of vertices such that every vertex in the graph is dominated by some element of D.
- Finding minimum dominating sets is an NP-complete problem with many applications (e.g., resource allocation).
- The *domination ratio* $\bar{\gamma}(\mathbb{Z}, S)$ of the integer distance graph $\Gamma(\mathbb{Z}, S)$ is the infimum of $\delta(D)$ over all dominating sets D of $\Gamma(\mathbb{Z}, S)$.
- We have $\mathbb{Z} = S \oplus D$ if and only if every vertex in $\Gamma(\mathbb{Z}, S)$ is dominated by exactly one element of D, i.e., D is an *efficient dominating set*.
- Efficient dominating sets in a finite Cayley graph has been studied by Chelvam and Mutharasu, Dejter and Serra, and others.

Period sets

Definition

A set $D \subseteq \mathbb{Z}$ is *periodic* if there exists a positive integer p such that

$$D \cap [ip+1, ip+p] = \{ip + j : j \in D \cap [1, p]\}, \quad \forall i \in \mathbb{Z}.$$

The smallest p (not necessarily prime) is called the *period* of D.

Period sets

Definition

A set $D \subseteq \mathbb{Z}$ is *periodic* if there exists a positive integer p such that

$$D \cap [ip+1, ip+p] = \{ip+j : j \in D \cap [1,p]\}, \quad \forall i \in \mathbb{Z}.$$

The smallest p (not necessarily prime) is called the *period* of D.

Proposition (H. 2019)

Let S be a finite subset of $\mathbb{Z} \setminus \{0\}$. The following results hold.

- *The domination ratio of $\Gamma(\mathbb{Z}, S)$ is the density $\delta(D) = |D \cap [1,p]|/p$ of some periodic dominating set D with period p.*

Period sets

Definition

A set $D \subseteq \mathbb{Z}$ is *periodic* if there exists a positive integer p such that

$$D \cap [ip+1, ip+p] = \{ip+j : j \in D \cap [1,p]\}, \quad \forall i \in \mathbb{Z}.$$

The smallest p (not necessarily prime) is called the *period* of D.

Proposition (H. 2019)

Let S be a finite subset of $\mathbb{Z} \setminus \{0\}$. The following results hold.

- *The domination ratio of $\Gamma(\mathbb{Z}, S)$ is the density $\delta(D) = |D \cap [1,p]|/p$ of some periodic dominating set D with period p.*
- *By reduction modulo p, we get a minimum dominating set $D \cap [1,p]$ for the circulant graph $\Gamma(\mathbb{Z}_p, S_p)$.*

Period sets

Definition

A set $D \subseteq \mathbb{Z}$ is *periodic* if there exists a positive integer p such that

$$D \cap [ip+1, ip+p] = \{ip+j : j \in D \cap [1,p]\}, \quad \forall i \in \mathbb{Z}.$$

The smallest p (not necessarily prime) is called the *period* of D.

Proposition (H. 2019)

Let S be a finite subset of $\mathbb{Z} \setminus \{0\}$. The following results hold.

- *The domination ratio of $\Gamma(\mathbb{Z}, S)$ is the density $\delta(D) = |D \cap [1,p]|/p$ of some periodic dominating set D with period p.*
- *By reduction modulo p, we get a minimum dominating set $D \cap [1,p]$ for the circulant graph $\Gamma(\mathbb{Z}_p, S_p)$.*
- *The graph $\Gamma(\mathbb{Z}, S)$ has an efficient dominating set if and only if its domination ratio is $1/(|S|+1)$.*

The case $|S| = 2$

Theorem (H. 2019)

If $k \in \mathbb{Z}$ *then* $\bar{\gamma}(\mathbb{Z}, \{1, 3k+2\}) = 1/3$. *If* k *is a positive integer then*

$$\bar{\gamma}(\mathbb{Z}, \{1, 3k+1\}) = \bar{\gamma}(\mathbb{Z}, \{1, -3k\}) = (k+1)/(3k+2),$$

$$\bar{\gamma}(\mathbb{Z}, \{1, 3k\}) = \bar{\gamma}(\mathbb{Z}, \{1, -3k+1\}) = 2k/(6k-1).$$

The case $|S| = 2$

Theorem (H. 2019)

If $k \in \mathbb{Z}$ then $\bar{\gamma}(\mathbb{Z}, \{1, 3k+2\}) = 1/3$. If k is a positive integer then

$$\bar{\gamma}(\mathbb{Z}, \{1, 3k+1\}) = \bar{\gamma}(\mathbb{Z}, \{1, -3k\}) = (k+1)/(3k+2),$$

$$\bar{\gamma}(\mathbb{Z}, \{1, 3k\}) = \bar{\gamma}(\mathbb{Z}, \{1, -3k+1\}) = 2k/(6k-1).$$

Corollary (H. 2019)

Let $\gamma(\mathbb{Z}_p, S)$ be the domination number of $\Gamma(\mathbb{Z}_p, S)$. For $k > 0$ we have $\gamma(\mathbb{Z}_{3k+2}, \{\pm 1\}) = \gamma(\mathbb{Z}_{3k+2}, \{1, 2\}) = k+1$ and $\gamma(\mathbb{Z}_{6k-1}, \{1, 3k\}) = 2k$.

The case $|S| = 2$

Theorem (H. 2019)

If $k \in \mathbb{Z}$ then $\bar{\gamma}(\mathbb{Z}, \{1, 3k+2\}) = 1/3$. If k is a positive integer then

$$\bar{\gamma}(\mathbb{Z}, \{1, 3k+1\}) = \bar{\gamma}(\mathbb{Z}, \{1, -3k\}) = (k+1)/(3k+2),$$

$$\bar{\gamma}(\mathbb{Z}, \{1, 3k\}) = \bar{\gamma}(\mathbb{Z}, \{1, -3k+1\}) = 2k/(6k-1).$$

Corollary (H. 2019)

Let $\gamma(\mathbb{Z}_p, S)$ be the domination number of $\Gamma(\mathbb{Z}_p, S)$. For $k > 0$ we have $\gamma(\mathbb{Z}_{3k+2}, \{\pm 1\}) = \gamma(\mathbb{Z}_{3k+2}, \{1, 2\}) = k + 1$ and $\gamma(\mathbb{Z}_{6k-1}, \{1, 3k\}) = 2k$.

Problem

Determine $\bar{\gamma}(\mathbb{Z}, \{s, t\})$ where $s \nmid t$. (If $s \mid t$ then $\bar{\gamma}(\mathbb{Z}, \{s, t\}) = \bar{\gamma}(1, t/s)$.)

The case $|S| > 2$

Theorem (H. 2019+)

Let d and s be integers with $d \geq 2$ and $s \notin [0, d-2]$. Write $s = dk + e - 1$ or $s = -dk + d - e - 1$ for some integers $k \geq 1$ and $e \in \{1, \ldots, d-1\}$. Then $\Gamma(\mathbb{Z}, \{1, 2, \ldots, d-2, s\})$ has domination ratio

$$\begin{aligned}
&\bar{\gamma}(\mathbb{Z}, \{1, 2, \ldots, d-2, s\}) \\
&= \min\left\{ \frac{k+1}{dk+e}, \frac{2k+e-1}{2dk-d+2e}, \frac{1}{d-1} \right\} \\
&= \begin{cases} (k+1)/(dk+e) & \text{if } e \geq 2,\ d \leq k+e+1 \\ (2k+e-1)/(2dk-d+2e) & \text{if } e = 1,\ d \leq 2k+2 \\ 1/(d-1) & \text{otherwise.} \end{cases}
\end{aligned}$$

This ratio is achieved by a dominating set with block structure $(d^k, e)^\infty$, $(d^{k-1}, d+e, d^{k-1}, 1^e)^\infty$, or $(d-1)^\infty$.

Some corollaries

Corollary (H. 2019+)

If $s = 4k$ *or* $-4k + 2$ *then* $\bar{\gamma}(\mathbb{Z}, \{1, 2, s\}) = 2k/(8k - 2)$.
If $s = 4k + 1$ *or* $-4k + 1$ $(k > 0)$ *then* $\bar{\gamma}(\mathbb{Z}, \{1, 2, s\}) = (k + 1)/(4k + 2)$.
If $s = 4k + 2$ *or* $-4k$ $(k > 0)$ *then* $\bar{\gamma}(\mathbb{Z}, \{1, 2, s\}) = (k + 1)/(4k + 3)$.

Some corollaries

Corollary (H. 2019+)

If $s = 4k$ *or* $-4k+2$ *then* $\bar{\gamma}(\mathbb{Z}, \{1,2,s\}) = 2k/(8k-2)$.
If $s = 4k+1$ *or* $-4k+1$ $(k>0)$ *then* $\bar{\gamma}(\mathbb{Z}, \{1,2,s\}) = (k+1)/(4k+2)$.
If $s = 4k+2$ *or* $-4k$ $(k>0)$ *then* $\bar{\gamma}(\mathbb{Z}, \{1,2,s\}) = (k+1)/(4k+3)$.

Corollary (H. 2019+)

Let d, s *be integers with* $d \geq 2$ *and* $s \notin [0, d-2]$. *Then there exists an efficient dominating set for* $\Gamma(\mathbb{Z}, \{1, 2, \ldots, d-2, s\})$ *if and only if* $d = 2$ *or* $s \equiv -1 \pmod{d}$.

Some corollaries

Corollary (H. 2019+)

If $s = 4k$ *or* $-4k+2$ *then* $\bar{\gamma}(\mathbb{Z}, \{1,2,s\}) = 2k/(8k-2)$.
If $s = 4k+1$ *or* $-4k+1$ $(k > 0)$ *then* $\bar{\gamma}(\mathbb{Z}, \{1,2,s\}) = (k+1)/(4k+2)$.
If $s = 4k+2$ *or* $-4k$ $(k > 0)$ *then* $\bar{\gamma}(\mathbb{Z}, \{1,2,s\}) = (k+1)/(4k+3)$.

Corollary (H. 2019+)

Let d, s *be integers with* $d \geq 2$ *and* $s \notin [0, d-2]$. *Then there exists an efficient dominating set for* $\Gamma(\mathbb{Z}, \{1, 2, \ldots, d-2, s\})$ *if and only if* $d = 2$ *or* $s \equiv -1 \pmod{d}$.

Corollary (H. 2019+)

Let $d \geq 2$, $k \geq 1$, *and* $e \geq 2$ *be integers. If* $d \leq k+e+1$ *then*
$\gamma(\mathbb{Z}_{dk+e}, \{-1, 1, 2, \ldots, d-2\}) = \gamma(\mathbb{Z}_{dk+e}, \{1, 2, \ldots, d-1\}) = k+1$.
If $d \leq 2k+2$ *then* $\gamma(\mathbb{Z}_{2dk-d+2}, \{1, 2, \ldots, d-2, dk\}) = 2k$.

Thank you!