Integer Tillings and Domination Ratio

Jia Huang

University of Nebraska at Kearney

E-mail address: huangj2@unk.edu

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A finite set $S \subseteq \mathbb{Z}$ tiles the integers if there exists a set $D \subseteq \mathbb{Z}$ such that $\mathbb{Z} = S \oplus D := \bigcup_{d \in D} (S + d)$. 

Example: $S = \{0, 1, 2\}$ tiles the integers, but $S = \{0, 1, 3\}$ does not.

Newman (1977) determined whether a given set $S$ with cardinality $|S| = p^a$ for some prime $p$ can tile the integers.

Coven and Meyerowitz (1999) generalized the result of Newman to the case $|S| = p^a q^b$ for two primes $p, q$.

Example: $S = \{1, 4, 8, 13\}$ tiles the integers, but $S = \{1, 4, 8, 15\}$ does not.

Observation: We may assume $0 \in S$, without loss of generality.
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Tiling the Integers without overlaps

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If $\mathbb{Z} = S \oplus D$, the density of $D$ must be $\delta(D) = 1/|S|$, where

$$\delta(D) := \liminf_{n \to \infty} \frac{|D \cap [-n, n]|}{2n + 1}.$$
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- If $S$ cannot tile $\mathbb{Z}$, we can still look for a set $D \subseteq \mathbb{Z}$ with minimum density $\delta(D)$ such that $\mathbb{Z} = \bigcup_{d \in D} (S + d)$. 

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- Such a set \( D \) is actually a minimum dominating set of the integer distance graph \( \Gamma(\mathbb{Z}, S) \), which is a Cayley graph with vertex set \( \mathbb{Z} \) and edge set \( \{(n, n + s) : n \in \mathbb{Z}, s \in S\} \).
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- The independence ratio of an integer distance graph is closely related to its chromatic number and has been extensively studied.
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A *dominating set* $D$ of a graph is a set of vertices such that every vertex in the graph is dominated by some element of $D$. 

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The domination ratio $\bar{\gamma}(\mathbb{Z}, S)$ of the integer distance graph $\Gamma(\mathbb{Z}, S)$ is the infimum of $\delta(D)$ over all dominating sets $D$ of $\Gamma(\mathbb{Z}, S)$. 
Domination Ratio of an integer distance graph

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- We have $\mathbb{Z} = S \oplus D$ if and only if every vertex in $\Gamma(\mathbb{Z}, S)$ is dominated by exactly one element of $D$, i.e., $D$ is an *efficient dominating set*. 
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Period sets

**Definition**

A set $D \subseteq \mathbb{Z}$ is *periodic* if there exists a positive integer $p$ such that

$$D \cap [ip + 1, ip + p] = \{ip + j : j \in D \cap [1, p]\}, \quad \forall i \in \mathbb{Z}.$$  

The smallest $p$ (not necessarily prime) is called the *period* of $D$.  

\[\text{Proposition (H. 2019)\hspace{1cm}}\]

Let $S$ be a finite subset of $\mathbb{Z}\{-0\}$. The following results hold.

- The domination ratio of $\Gamma(Z, S)$ is the density $\delta(D) = \frac{|D \cap [1, p]|}{p}$ of some periodic dominating set $D$ with period $p$.
- By reduction modulo $p$, we get a minimum dominating set $D \cap [1, p]$ for the circulant graph $\Gamma(Z_p, S_p)$.
- The graph $\Gamma(Z, S)$ has an efficient dominating set if and only if its domination ratio is $\frac{1}{|S| + 1}$.  

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Let $S$ be a finite subset of $\mathbb{Z} \setminus \{0\}$. The following results hold.

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The case $|S| = 2$

**Theorem (H. 2019)**

If $k \in \mathbb{Z}$ then $\bar{\gamma}(\mathbb{Z}, \{1, 3k + 2\}) = 1/3$. If $k$ is a positive integer then

$$\bar{\gamma}(\mathbb{Z}, \{1, 3k + 1\}) = \bar{\gamma}(\mathbb{Z}, \{1, -3k\}) = (k + 1)/(3k + 2),$$

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**Corollary (H. 2019)**

Let $\gamma(\mathbb{Z}_p, S)$ be the domination number of $\Gamma(\mathbb{Z}_p, S)$. For $k > 0$ we have $\gamma(\mathbb{Z}_{3k+2}, \{\pm 1\}) = \gamma(\mathbb{Z}_{3k+2}, \{1, 2\}) = k + 1$ and $\gamma(\mathbb{Z}_{6k-1}, \{1, 3k\}) = 2k$. 
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**Problem**

Determine $\bar{\gamma}(\mathbb{Z}, \{s, t\})$ where $s \nmid t$. (If $s \mid t$ then $\bar{\gamma}(\mathbb{Z}, \{s, t\}) = \bar{\gamma}(1, t/s)$.)
The case $|S| > 2$

**Theorem (H. 2019+)**

Let $d$ and $s$ be integers with $d \geq 2$ and $s \notin [0, d - 2]$. Write $s = dk + e - 1$ or $s = -dk + d - e - 1$ for some integers $k \geq 1$ and $e \in \{1, \ldots, d - 1\}$. Then $\Gamma(\mathbb{Z}, \{1, 2, \ldots, d - 2, s\})$ has domination ratio

$$\bar{\gamma}(\mathbb{Z}, \{1, 2, \ldots, d - 2, s\}) = \min \left\{ \frac{k + 1}{dk + e}, \frac{2k + e - 1}{2dk - d + 2e}, \frac{1}{d - 1} \right\}$$

$$= \begin{cases} 
    \frac{(k + 1)}{(dk + e)} & \text{if } e \geq 2, \ d \leq k + e + 1 \\
    \frac{(2k + e - 1)}{(2dk - d + 2e)} & \text{if } e = 1, \ d \leq 2k + 2 \\
    \frac{1}{(d - 1)} & \text{otherwise.}
\end{cases}$$

This ratio is achieved by a dominating set with block structure $(d^k, e)^\infty$, $(d^{k-1}, d + e, d^{k-1}, 1^e)^\infty$, or $(d - 1)^\infty$. 

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Some corollaries

Corollary (H. 2019+)

If \( s = 4k \) or \(-4k + 2\) then \( \bar{\gamma}(\mathbb{Z}, \{1, 2, s\}) = 2k/(8k - 2) \).

If \( s = 4k + 1 \) or \(-4k + 1\) (\( k > 0 \)) then \( \bar{\gamma}(\mathbb{Z}, \{1, 2, s\}) = (k + 1)/(4k + 2) \).

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#### Corollary (H. 2019+)

Let $d, s$ be integers with $d \geq 2$ and $s \notin [0, d - 2]$. Then there exists an efficient dominating set for $\Gamma(\mathbb{Z}, \{1, 2, \ldots, d - 2, s\})$ if and only if $d = 2$ or $s \equiv -1 \pmod{d}$.
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**Corollary (H. 2019+)**

Let $d \geq 2$, $k \geq 1$, and $e \geq 2$ be integers. If $d \leq k + e + 1$ then $\gamma(\mathbb{Z}_{dk+e}, \{-1, 1, 2, \ldots, d - 2\}) = \gamma(\mathbb{Z}_{dk+e}, \{1, 2, \ldots, d - 1\}) = k + 1$.
If $d \leq 2k + 2$ then $\gamma(\mathbb{Z}_{2dk-d+2}, \{1, 2, \ldots, d - 2, dk\}) = 2k$. 
Thank you!