An improved upper bound for the bondage number of graphs on surfaces

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This is joint work with Jian Shen.

June 17, 2018
Domination in graphs

**Definition**
A *dominating* set for a graph $G$ is a subset $D$ of vertices such that every vertex not in $D$ is adjacent to some vertex in $D$.

**Definition**
The *domination number* $\gamma(G)$ of $G$ is the cardinality of a minimum dominating set of $G$.

**Example**
$\gamma(K_n) = 1$, $\gamma(P_n) = \gamma(C_n) = \lceil n/3 \rceil$.

**Proposition**
It is NP-hard to find a minimum dominating set for a graph $G$. 
Applications and link failure

- There are many applications of domination in networks, such as resource allocation.
- In reality the structure of a network might change.
- An example is link failure (due to various reasons).
- The domination number of a graph weakly will increase when some edges are deleted.
- When will the domination number strictly increase?
The bondage number

**Definition (Fink, Jacobson, Kinch, and Roberts, 1990)**

The *bondage number* $b(G)$ of a graph $G$ is defined as the smallest number of edges whose removal from $G$ results in a graph with larger domination number.

**Example**

$b(K_n) = \lceil n/2 \rceil$, $b(C_n) = b(P_n) + 1$, and

$$b(P_n) = \begin{cases} 2, & n \equiv 1 \mod 3, \\ 1, & \text{otherwise}. \end{cases}$$
<table>
<thead>
<tr>
<th>Proposition (Hu and Xu 2012)</th>
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<tbody>
<tr>
<td><em>It is NP-hard to determine the bondage number</em> $b(G)$.</td>
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<table>
<thead>
<tr>
<th>Lemma (Hartnell and Rall 1994)</th>
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<tbody>
<tr>
<td>For any edge $uv \in E(G)$, $b(G) \leq d(u) + d(v) - 1 -</td>
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<table>
<thead>
<tr>
<th>Corollary</th>
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<tbody>
<tr>
<td>For any graph $G$ with maximum degree $\Delta(G)$ and minimum degree $\delta(G)$, one has $b(G) \leq \Delta(G) + \delta(G) - 1$.</td>
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</tbody>
</table>
Conjectures

Conjecture (Teschner 1995)

For any graph $G$, $b(G) \leq \frac{3}{2} \Delta(G)$.

Conjecture (Dunbar-Haynes-Teschner-Volkmann 1998)

For any planar graph $G$, $b(G) \leq \Delta(G) + 1$. 
Early results

Theorem (Kang and Yuan 2000)

For any planar graph $G$, $b(G) \leq \min\{\Delta(G) + 2; 8\}$.

Theorem (Carlson and Develin 2006)

Let $G$ be a graph embedded on a torus. Then $b(G) \leq \Delta(G) + 3$.

Remark

The method of Carlson and Develin provides a simpler proof for the result of Kang and Yuan.
Classification Theorem for Surfaces

Any surface $S$ is homeomorphic to either of the following surfaces:
- $S_h$ obtained from a sphere by adding $h \geq 0$ handles,
- $N_k$ obtained from a sphere by adding $k \geq 1$ crosscaps.

Definition

A surface $S$ is an orientable surface of genus $h$ if $S \cong S_h$, or a non-orientable surface of genus $k$ if $S \cong N_k$.

Example

The torus, the projective plane, and the Klein bottle are homeomorphic to $S_1$, $N_1$, and $N_2$, respectively.
Theorem (Gagarin and Zverovich)

Let $G$ be a graph embeddable on an orientable surface of genus $h$ and a non-orientable surface of genus $k$. Then

$$b(G) \leq \min\{\Delta(G) + h + 2, \Delta(G) + k + 1\}.$$ 

Remark (Gagarin and Zverovich)

When $h$ and $k$ are large one can achieve better results, such as

$$b(G) \leq \Delta(G) + \begin{cases} h + 1, & \text{if } h \geq 8, \\ h, & \text{if } h \geq 11, \\ k, & \text{if } k \geq 3, \\ k - 1, & \text{if } k \geq 6. \end{cases}$$
Euler characteristic

**Definition**

The *Euler characteristic* of a surface $S$ is defined as

$$
\chi(S) = \begin{cases} 
2 - 2h, & S \cong S_h, \\
2 - k, & S \cong N_k.
\end{cases}
$$

**Example**

<table>
<thead>
<tr>
<th>$S$</th>
<th>$S_0$</th>
<th>$S_1$</th>
<th>$S_2$</th>
<th>$N_1$</th>
<th>$N_2$</th>
<th>$N_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi$</td>
<td>2</td>
<td>0</td>
<td>-2</td>
<td>1</td>
<td>0</td>
<td>-1</td>
</tr>
</tbody>
</table>

**Euler’s Formula**

*If a graph $G$ admits a (2-cell) embedding on a surface $S$ with $V(G) = \{\text{vertices}\}$, $E(G) = \{\text{edges}\}$, $F(G) = \{\text{faces}\}$, then*

$$
|V(G)| - |E(G)| + |F(G)| = \chi(S).
$$
An improved upper bound

Theorem (H. and Shen)

Let $G$ be a graph embedded on a surface whose Euler characteristic $\chi$ is as large as possible. Assume $\chi \leq 0$. Then

$$b(G) \leq \Delta(G) + \lfloor t \rfloor$$

where $t = t(\chi)$ is the largest real root of

$$z^3 + z^2 + (3\chi - 8)z + 9\chi - 12.$$ 

Remark

Our theorem implies the earlier result of Gagarin and Zverovich.
Explicit values

Remark

We have \( t = t(\chi) = \frac{1}{3} (D + (25 - 9\chi)/D - 1) \) where

\[
D = \left(9\sqrt{9\chi^3 + 69\chi^2 - 125\chi - 108\chi + 125}\right)^{\frac{1}{3}}.
\]

Example

<table>
<thead>
<tr>
<th>\chi</th>
<th>0</th>
<th>-1</th>
<th>-2</th>
<th>-3</th>
<th>-4</th>
<th>-5</th>
<th>-6</th>
<th>-7</th>
<th>-8</th>
<th>-9</th>
<th>-10</th>
</tr>
</thead>
<tbody>
<tr>
<td>[t]</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>6</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>GZ</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>5</td>
<td>6</td>
<td>6</td>
<td>8</td>
<td>7</td>
<td>10</td>
<td>8</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>\chi</th>
<th>-11</th>
<th>-12</th>
<th>-13</th>
<th>-14</th>
<th>-15</th>
<th>-16</th>
<th>-17</th>
<th>-18</th>
<th>-19</th>
<th>-20</th>
<th>-21</th>
</tr>
</thead>
<tbody>
<tr>
<td>[t]</td>
<td>7</td>
<td>7</td>
<td>7</td>
<td>7</td>
<td>7</td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>9</td>
</tr>
<tr>
<td>GZ</td>
<td>12</td>
<td>9</td>
<td>14</td>
<td>9</td>
<td>16</td>
<td>10</td>
<td>18</td>
<td>11</td>
<td>20</td>
<td>11</td>
<td>22</td>
</tr>
</tbody>
</table>
Corollary (H. and Shen)

Let $G$ be a graph embedded on a surface whose Euler characteristic $\chi$ is as large as possible. If $\chi \leq 0$ then

$$b(G) \leq \Delta(G) + 1 + \lfloor \sqrt{4 - 3\chi} \rfloor.$$ 

Remark

This corollary is implied by the previous theorem, but also asymptotically equivalent to it:

$$\lim_{\chi \to -\infty} \frac{t(\chi)}{1 + \sqrt{4 - 3\chi}} = 1.$$
Graphs with large girth

**Definition**

The *girth* \( g(G) \) of a graph \( G \) is the length of the shortest cycle in \( G \). If \( G \) has no cycle then \( g(G) = \infty \) (and \( b(G) \leq 2 \)).

**Theorem (H. and Shen)**

Let \( G \) be a graph embedded on a surface whose Euler characteristic \( \chi \) is as large as possible. If \( \chi \leq 0 \) and \( g = g(G) < \infty \), then

\[
b(G) \leq \Delta(G) + \left\lceil \frac{2 + \sqrt{g^2 - g(g - 2)\chi}}{(g - 2)} \right\rceil.
\]

In particular, if \( G \) is triangle-free, then

\[
b(G) \leq \Delta(G) + 1 + \left\lceil \sqrt{4 - 2\chi} \right\rceil.
\]
Theorem (Gagarin and Zverovich)

Let $G$ be a connected graph 2-cell embeddable on an orientable surface of genus $h \geq 1$ and a non-orientable surface of genus $k \geq 1$. Let $n = |V(G)|$. Then

- $b(G) \leq \Delta(G) + \lceil \ln^2 h \rceil + 3$ if $n \geq h$,
- $b(G) \leq \Delta(G) + \lceil \ln h \rceil + 3$ if $n \geq h^{1.9}$,
- $b(G) \leq \Delta(G) + 4$ if $n \geq h^{2.5}$,
- $b(G) \leq \Delta(G) + \lceil \ln^2 k \rceil + 2$ if $n \geq k/6$,
- $b(G) \leq \Delta(G) + \lceil \ln k \rceil + 3$ if $n \geq k^{1.6}$,
- $b(G) \leq \Delta(G) + 3$ if $n \geq k^2$. 
Theorem (H. and Shen)

Let $G$ be a connected graph on a surface whose Euler characteristic $\chi$ is as large as possible. Let $n = |V(G)|$ and assume $\chi \leq 0$. Then

$$b(G) \leq \Delta(G) + \left[ \frac{1}{2} - \frac{3\chi}{n} + \sqrt{\frac{25}{4} - \frac{21\chi}{n} + \frac{9\chi^2}{n^2}} \right].$$

In particular, we have

- $b(G) \leq \Delta(G) + 9$ if $n \geq -\chi$,
- $b(G) \leq \Delta(G) + 6$ if $n \geq -2\chi$,
- $b(G) \leq \Delta(G) + 5$ if $n \geq -3\chi$,
- $b(G) \leq \Delta(G) + 4$ if $n \geq -4\chi$,
- $b(G) \leq \Delta(G) + 3$ if $n \geq -8\chi$. 

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**Theorem (H. and Shen)**

Let $G$ be a connected graph embedded on a surface whose Euler characteristic $\chi$ is as large as possible. Suppose that $m = |E(G)| > -3\chi \geq 0$. Then

$$b(G) \leq \Delta(G) + \left[ 3 - \frac{18\chi}{m + 3\chi} \right].$$

In particular, we have

- $b(G) \leq \Delta(G) + 8$ if $m > -6\chi$,
- $b(G) \leq \Delta(G) + 7$ if $m > -6.6\chi$,
- $b(G) \leq \Delta(G) + 6$ if $m > -7.5\chi$,
- $b(G) \leq \Delta(G) + 5$ if $m > -9\chi$,
- $b(G) \leq \Delta(G) + 4$ if $m > -12\chi$,
- $b(G) \leq \Delta(G) + 3$ if $m > -21\chi$. 

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Thank you!