An improved upper bound for the bondage number of graphs on surfaces

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Domination in graphs

Definition

A *dominating* set for a graph G is a subset D of vertices such that every vertex not in D is adjacent to some vertex in D.

Definition

The *domination number* $\gamma(G)$ of G is the cardinality of a minimum dominating set of G.

Example

$\gamma(K_n) = 1$, $\gamma(P_n) = \gamma(C_n) = \lceil n/3 \rceil$.

Proposition

It is NP-hard to find a minimum dominating set for a graph G.

Applications and link failure

- There are many applications of domination in networks, such as resource allocation.
- In reality the structure of a network might change.
- An example is link failure (due to various reasons).
- The domination number of a graph weakly will increase when some edges are deleted.
- When will the domination number strictly increase?

The bondage number

Definition (Fink, Jacobson, Kinch, and Roberts, 1990)

The *bondage number* $b(G)$ of a graph G is defined as the smallest number of edges whose removal from G results in a graph with larger domination number.

Example

$b(K_n) = \lceil n/2 \rceil$, $b(C_n) = b(P_n) + 1$, and

$$b(P_n) = \begin{cases} 2, & n \equiv 1 \mod 3, \\ 1, & \text{otherwise.} \end{cases}$$

Bounds for the bondage number

Proposition (Hu and Xu 2012)

It is NP-hard to determine the bondage number $b(G)$.

Lemma (Hartnell and Rall 1994)

For any edge $uv \in E(G)$, $b(G) \leq d(u) + d(v) - 1 - |N(u) \cap N(v)|$.

Corollary

For any graph G with maximum degree $\Delta(G)$ and minimum degree $\delta(G)$, one has $b(G) \leq \Delta(G) + \delta(G) - 1$.

Conjectures

Conjecture (Teschner 1995)

For any graph G, $b(G) \leq \frac{3}{2}\Delta(G)$.

Conjecture (Dunbar-Haynes-Teschner-Volkmann 1998)

For any planar graph G, $b(G) \leq \Delta(G) + 1$.

Early results

Theorem (Kang and Yuan 2000)

For any planar graph G, $b(G) \leq \min\{\Delta(G) + 2; 8\}$.

Theorem (Carlson and Develin 2006)

Let G be a graph embedded on a torus. Then $b(G) \leq \Delta(G) + 3$.

Remark

The method of Carlson and Develin provides a simpler proof for the result of Kang and Yuan.

Graph embedding on surfaces

Classification Theorem for Surfaces

Any surface S is homeomorphic to either of the following surfaces:

- *S_h obtained from a sphere by adding $h \geq 0$ handles,*
- *N_k obtained from a sphere by adding $k \geq 1$ crosscaps.*

Definition

A surface S is an *orientable surface of genus h* if $S \cong S_h$, or a *non-orientable surface of genus k* if $S \cong N_k$.

Example

The torus, the projective plane, and the Klein bottle are homeomorphic to S_1, N_1, and N_2, respectively.

Upper bound in terms of genera

Theorem (Gagarin and Zverovich)

Let G be a graph embeddable on an orientable surface of genus h and a non-orientable surface of genus k. Then

$$b(G) \leq \min\{\Delta(G) + h + 2, \Delta(G) + k + 1\}.$$

Remark (Gagarin and Zverovich)

When h and k are large one can achieve better results, such as

$$b(G) \leq \Delta(G) + \begin{cases} h+1, & \text{if } h \geq 8, \\ h, & \text{if } h \geq 11, \\ k, & \text{if } k \geq 3, \\ k-1, & \text{if } k \geq 6. \end{cases}$$

Euler characteristic

Definition

The *Euler characteristic* of a surface S is defined as

$$\chi(S) = \begin{cases} 2-2h, & S \cong S_h, \\ 2-k, & S \cong N_k. \end{cases}$$

Example

S	S_0	S_1	S_2	N_1	N_2	N_3
χ	2	0	-2	1	0	-1

Euler's Formula

If a graph G admits a (2-cell) embedding on a surface S with $V(G) = \{vertices\}$, $E(G) = \{edges\}$, $F(G) = \{faces\}$, then

$$|V(G)| - |E(G)| + |F(G)| = \chi(S).$$

An improved upper bound

Theorem (H. and Shen)

Let G be a graph embedded on a surface whose Euler characteristic χ is as large as possible. Assume $\chi \leq 0$. Then

$$b(G) \leq \Delta(G) + \lfloor t \rfloor$$

where $t = t(\chi)$ is the largest real root of

$$z^3 + z^2 + (3\chi - 8)z + 9\chi - 12.$$

Remark

Our theorem implies the earlier result of Gagarin and Zverovich.

Explicit values

Remark

We have $t = t(\chi) = \frac{1}{3}\left(D + (25 - 9\chi)/D - 1\right)$ where

$$D = \left(9\sqrt{9\chi^3 + 69\chi^2 - 125\chi} - 108\chi + 125\right)^{\frac{1}{3}}.$$

Example

χ	0	-1	-2	-3	-4	-5	-6	-7	-8	-9	-10
$\lfloor t \rfloor$	3	3	4	4	4	5	5	5	6	6	6
GZ	3	3	4	5	5	6	6	8	7	10	8

χ	-11	-12	-13	-14	-15	-16	-17	-18	-19	-20	-21
$\lfloor t \rfloor$	7	7	7	7	7	8	8	8	8	8	9
GZ	12	9	14	9	16	10	18	11	20	11	22

Approximation

Corollary (H. and Shen)

Let G be a graph embedded on a surface whose Euler characteristic χ is as large as possible. If $\chi \leq 0$ then

$$b(G) \leq \Delta(G) + 1 + \lfloor \sqrt{4 - 3\chi} \rfloor.$$

Remark

This corollary is implied by the previous theorem, but also asymptotically equivalent to it:

$$\lim_{\chi \to -\infty} \frac{t(\chi)}{1 + \sqrt{4 - 3\chi}} = 1.$$

Graphs with large girth

Definition

The *girth* $g(G)$ of a graph G is the length of the shortest cycle in G. If G has no cycle then $g(G) = \infty$ (and $b(G) \leq 2$).

Theorem (H. and Shen)

Let G be a graph embedded on a surface whose Euler characteristic χ is as large as possible. If $\chi \leq 0$ and $g = g(G) < \infty$, then

$$b(G) \leq \Delta(G) + \left\lfloor \frac{2 + \sqrt{g^2 - g(g-2)\chi}}{(g-2)} \right\rfloor .$$

In particular, if G is triangle-free, then

$$b(G) \leq \Delta(G) + 1 + \left\lfloor \sqrt{4 - 2\chi} \right\rfloor .$$

Graphs with large order: first result

Theorem (Gagarin and Zverovich)

Let G be a connected graph 2-cell embeddable on an orientable surface of genus $h \geq 1$ and a non-orientable surface of genus $k \geq 1$. Let $n = |V(G)|$. Then

- $b(G) \leq \Delta(G) + \lceil \ln^2 h \rceil + 3$ *if* $n \geq h$,
- $b(G) \leq \Delta(G) + \lceil \ln h \rceil + 3$ *if* $n \geq h^{1.9}$,
- $b(G) \leq \Delta(G) + 4$ *if* $n \geq h^{2.5}$,
- $b(G) \leq \Delta(G) + \lceil \ln^2 k \rceil + 2$ *if* $n \geq k/6$,
- $b(G) \leq \Delta(G) + \lceil \ln k \rceil + 3$ *if* $n \geq k^{1.6}$,
- $b(G) \leq \Delta(G) + 3$ *if* $n \geq k^2$.

Graphs with large order: improvement

Theorem (H. and Shen)

Let G be a connected graph on a surface whose Euler characteristic χ is as large as possible. Let $n = |V(G)|$ and assume $\chi \leq 0$. Then

$$b(G) \leq \Delta(G) + \left\lfloor \frac{1}{2} - \frac{3\chi}{n} + \sqrt{\frac{25}{4} - \frac{21\chi}{n} + \frac{9\chi^2}{n^2}} \right\rfloor .$$

In particular, we have

- $b(G) \leq \Delta(G) + 9$ *if* $n \geq -\chi$,
- $b(G) \leq \Delta(G) + 6$ *if* $n \geq -2\chi$,
- $b(G) \leq \Delta(G) + 5$ *if* $n \geq -3\chi$,
- $b(G) \leq \Delta(G) + 4$ *if* $n \geq -4\chi$,
- $b(G) \leq \Delta(G) + 3$ *if* $n \geq -8\chi$.

Graphs with large size

Theorem (H. and Shen)

Let G be a connected graph embedded on a surface whose Euler characteristic χ is as large as possible. Suppose that $m = |E(G)| > -3\chi \geq 0$. Then

$$b(G) \leq \Delta(G) + \left\lfloor 3 - \frac{18\chi}{m+3\chi} \right\rfloor .$$

In particular, we have

- $b(G) \leq \Delta(G) + 8$ *if* $m > -6\chi$,
- $b(G) \leq \Delta(G) + 7$ *if* $m > -6.6\chi$,
- $b(G) \leq \Delta(G) + 6$ *if* $m > -7.5\chi$,
- $b(G) \leq \Delta(G) + 5$ *if* $m > -9\chi$,
- $b(G) \leq \Delta(G) + 4$ *if* $m > -12\chi$,
- $b(G) \leq \Delta(G) + 3$ *if* $m > -21\chi$.

Thank you!