Arithmetic without parentheses

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Introduction

Example

48 ÷ 2(9 + 3) = ?

Solution (Google)

\[
\begin{align*}
48 & ÷ 2(9 + 3) \\
& = 48 ÷ 2(12) \\
& = 24 \times 12 \\
& = 288
\end{align*}
\]

Solution (PEMDAS)

\[
\begin{align*}
48 & ÷ 2(9 + 3) \\
& = 48 ÷ 2(12) \\
& = 48 ÷ 24 \\
& = 2
\end{align*}
\]

Conclusion

An arithmetic expression without parentheses is ambiguous.

Problem

How ambiguous is it?
Subtraction

Example (1)

\[(a - b) = a - b\]

Example (2)

\[(a - b) - c = a - b - c\]
\[a - (b - c) = a - b + c\]

Example (3)

\[((a - b) - c) - d = a - b - c - d\]
\[(a - b) - (c - d) = a - b - c + d\]
\[(a - (b - c)) - d = a - b + c - d\]
\[a - ((b - c) - d) = a - b + c + d\]
\[a - (b - (c - d)) = a - b + c - d\]

Problem

C. How many ways to insert parentheses in \(x_0 - x_1 - x_2 - \cdots - x_n\)?

D. How many distinct functions can be obtained from this expression?

Solution (with computer aids)

| \(n\) | 1 | 2 | 3 | 4 | 5 | 6 | \(
\cdots\) |
|---|---|---|---|---|---|---|---|
| \(C_n\) | 1 | 2 | 5 | 14 | 42 | 132 | \(
\cdots\) |
| \(D_n\) | 1 | 2 | 4 | 8 | 16 | 32 | \(
\cdots\) |

Theorem

- \(C_n = \frac{1}{n+1} \binom{2n}{n}\). (see wiki)
- \(D_n = 2^{n-1}\). (Exercise)
A variation of subtraction

Definition (1)

\[(a * b) = -a - b\]

Example (2)

\[(a * b) * c = a + b - c\]
\[a * (b * c) = -a + b + c\]

Example (3)

\[((a * b) * c) * d = -a - b + c - d\]
\[(a * b) * (c * d) = a + b + c + d\]
\[(a * (b * c)) * d = a - b - c - d\]
\[a * ((b * c) * d) = -a - b - c + d\]
\[a * (b * (c * d)) = -a + b - c - d\]

Problem

C’. How many ways to insert parentheses in \(x_0 * x_1 * x_2 * \cdots * x_n\)?

D’. How many distinct functions can be obtained from this expression?

Solution (with computer aids)

<table>
<thead>
<tr>
<th>(n)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>(\cdots)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(C_n)</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>14</td>
<td>42</td>
<td>132</td>
<td>(\cdots)</td>
</tr>
<tr>
<td>(D_n)</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>10</td>
<td>21</td>
<td>42</td>
<td>(\cdots)</td>
</tr>
</tbody>
</table>

Problem

- \(C_n' = \frac{1}{n+1} \binom{2n}{n} = C_n\).
- \(D_n' = ?\)
Binary Trees

Definition

- A (full) binary tree is a structure in which each node has two children.
- The left (right) depth of a node is the number of left (right) steps from the root down to this node; \( \text{depth} = \text{left depth} + \text{right depth} \).

Fact

Ways to parenthesize \( x_0 \ast x_1 \ast \cdots \ast x_n \) \( \Leftrightarrow \) binary trees with \( n + 1 \) leaves.

Example

<table>
<thead>
<tr>
<th>parenthesization</th>
<th>( a \ast ((b \ast c) \ast d) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a \ast b = -a + b )</td>
<td>( -a + b - c + d )</td>
</tr>
<tr>
<td>( a \ast b = a - b )</td>
<td>( a - b + c + d )</td>
</tr>
<tr>
<td>( a \ast b = -a - b )</td>
<td>( -a - b - c + d )</td>
</tr>
</tbody>
</table>

Example

| binary tree | \( a \)
|-------------|-----
| \( b \) | \( c \)
| \( d \) |
| \( \text{left depth} \) | (1, 2, 1, 0) |
| \( \text{right depth} \) | (0, 1, 2, 2) |
| \( \text{depth} \) | (1, 3, 3, 2) |
Example (assuming \( c = d \))

\[
\begin{align*}
((a-b)-c)-d &= a - b - c - d = a - b - 2c \\
(a-b)-(c-d) &= a - b - c + d = a - b \\
(a-(b-c))-d &= a - b + c - d = a - b \\
a-((b-c)-d) &= a - b + c + d = a - b + 2c \\
a-(b-(c-d)) &= a - b + c - d = a - b
\end{align*}
\]

Problem

- How many distinct functions can be obtained from \( x_0 - x_1 - \cdots - x_n \) when \( x_0, x_1, \ldots, x_n \) are not all distinct? (e.g., \( x_2 = x_5 = x_7, x_3 = x_4 \))
- What if we are allowed to permute \( x_0, x_1, \ldots, x_n \)?
- What if we are allowed to use other arithmetic operations (\( +, \times, \div \))?
- What if we are allowed to use other operations (e.g., \( a^b \))?
More Examples

Example (Maths24 or The 24 Game)

\[(3 + 4) + 5 \times 2 = 24, \quad ((3 \div 3) + 5) \times 4 = 24, \quad 1 \ ? \ 1 \ ? \ 1 \ ? \ 1 \ ? \ 1 \ ? \ \overline{?} = 24\]

Example (Clocks with 9 only; a mistake in the left one)