Arithmetic without parentheses

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Introduction

Example

$48 \div 2(9+3) = ?$

Solution (Google)

$$\begin{aligned} & 48 \div 2(9+3) \\ = \; & 48 \div 2(12) \\ = \; & 24 \times 12 \\ = \; & 288 \end{aligned}$$

Solution (PEMDAS)

$$\begin{aligned} & 48 \div 2(9+3) \\ = \; & 48 \div 2(12) \\ = \; & 48 \div 24 \\ = \; & 2 \end{aligned}$$

Conclusion

An arithmetic expression without parentheses is ambiguous.

Problem

How ambiguous is it?

Subtraction

Example (1)

$$(a-b) = a-b$$

Example (2)

$$(a-b)-c = a-b-c$$
$$a-(b-c) = a-b+c$$

Example (3)

$$((a-b)-c)-d = a-b-c-d$$
$$(a-b)-(c-d) = a-b-c+d$$
$$(a-(b-c))-d = a-b+c-d$$
$$a-((b-c)-d) = a-b+c+d$$
$$a-(b-(c-d)) = a-b+c-d$$

Problem

C. *How many ways to insert parentheses in* $x_0 - x_1 - x_2 - \cdots - x_n$*?*

D. *How many distinct functions can be obtained from this expression?*

Solution (with computer aids)

n	1	2	3	4	5	6	$\cdots$
C_n	1	2	5	14	42	132	$\cdots$
D_n	1	2	4	8	16	32	$\cdots$

Theorem

- $C_n = \frac{1}{n+1}\binom{2n}{n}$. *(see wiki)*
- $D_n = 2^{n-1}$. *(Exercise)*

A variation of subtraction

Definition (1)

$$(a * b) = -a - b$$

Example (2)

$$\begin{aligned}(a * b) * c &= a + b - c \\ a * (b * c) &= -a + b + c\end{aligned}$$

Example (3)

$$\begin{aligned}((a * b) * c) * d &= -a - b + c - d \\ (a * b) * (c * d) &= a + b + c + d \\ (a * (b * c)) * d &= a - b - c - d \\ a * ((b * c) * d) &= -a - b - c + d \\ a * (b * (c * d)) &= -a + b - c - d\end{aligned}$$

Problem

C'. *How many ways to insert parentheses in* $x_0 * x_1 * x_2 * \cdots * x_n$?

D'. *How many distinct functions can be obtained from this expression?*

Solution (with computer aids)

n	1	2	3	4	5	6	$\cdots$
C'_n	1	2	5	14	42	132	$\cdots$
D'_n	1	2	5	10	21	42	$\cdots$

Problem

- $C'_n = \frac{1}{n+1}\binom{2n}{n} = C_n$.
- $D'_n = ?$

Binary Trees

Definition

- A (full) binary tree is a structure in which each node has two children.
- The *left (right) depth* of a node is the number of left (right) steps from the root down to this node; *depth* = left depth + right depth.

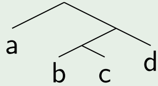
Fact

Ways to parenthesize $x_0 * x_1 * \cdots * x_n \Leftrightarrow$ *binary trees with* $n+1$ *leaves.*

Example

parenthesization	$a*((b*c)*d)$
$a*b=-a+b$	$-a+b-c+d$
$a*b=a-b$	$a-b+c+d$
$a*b=-a-b$	$-a-b-c+d$

Example

binary tree	
left depth	$(1,2,1,0)$
right depth	$(0,1,2,2)$
depth	$(1,3,3,2)$

Related Questions

Example (assuming $c = d$)

$$
\begin{aligned}
((a-b)-c)-d &= a-b-c-d &= a-b-2c \\
(a-b)-(c-d) &= a-b-c+d &= a-b \\
(a-(b-c))-d &= a-b+c-d &= a-b \\
a-((b-c)-d) &= a-b+c+d &= a-b+2c \\
a-(b-(c-d)) &= a-b+c-d &= a-b
\end{aligned}
$$

Problem

- *How many distinct functions can be obtained from $x_0 - x_1 - \cdots - x_n$ when $x_0, x_1 \ldots, x_n$ are not all distinct? (e.g., $x_2 = x_5 = x_7$, $x_3 = x_4$)*
- *What if we are allowed to permute $x_0, x_1, \ldots, x_n$?*
- *What if we are allowed to use other arithmetic operations ($+$, $\times$, $\div$)?*
- *What if we are allowed to use other operations (e.g., a^b)?*

More Examples

Example (Maths24 or The 24 Game)

$((3+4)+5)\times 2 = 24, \quad ((3\div 3)+5)\times 4 = 24, \quad 1 \ ? \ 1 \ ? \ 1 \ ? \ 1 \ ? \overset{?}{=} 24$

Example (Clocks with 9 only; a mistake in the left one)