

0-Hecke algebra action on the Stanley-Reisner ring of the Boolean algebra

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- The symmetric group and the 0-Hecke algebra.
- Actions on polynomials.
- Actions on Stanley-Reisner rings.
- Noncommutative Hall-Littlewood symmetric functions.
- Multivariate generating function of permutation statistics.

The Symmetric Groups and 0-Hecke algebra

- The symmetric group \mathfrak{S}_n is generated by the adjacent transpositions $s_i = (i, i + 1)$, $1 \leq i \leq n - 1$, with relations

$$\begin{cases} s_i^2 = 1, & 1 \leq i \leq n - 1, \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, & 1 \leq i \leq n - 2, \\ s_i s_j = s_j s_i, & |i - j| > 1. \end{cases}$$

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- The 0-Hecke algebra $H_n(0)$ is generated by the bubble-sorting operators π_1, \dots, π_{n-1} with relations

$$\begin{cases} \pi_i^2 = \pi_i, & 1 \leq i \leq n - 1, \\ \pi_i \pi_{i+1} \pi_i = \pi_{i+1} \pi_i \pi_{i+1}, & 1 \leq i \leq n - 2, \\ \pi_i \pi_j = \pi_j \pi_i, & |i - j| > 1. \end{cases}$$

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- Representations of \mathfrak{S}_n \leftrightarrow symmetric functions,
direct sum \leftrightarrow sum,
induction product \leftrightarrow product.

Compositions

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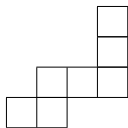
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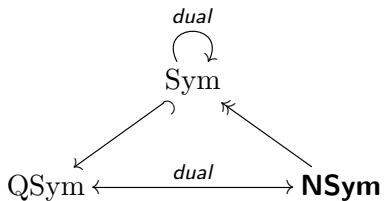
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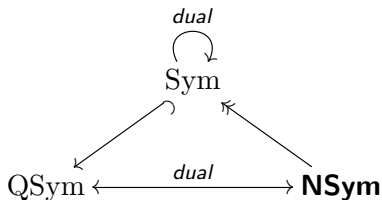
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$$\alpha = (2, 3, 1, 1) \models 7$$
$$D(\alpha) = \{2, 5, 6\} \subseteq [6]$$

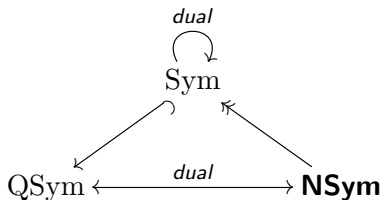
Sym, QSym, and NSym



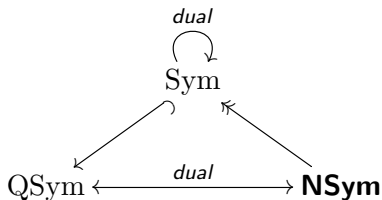


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- **NSym** (noncommutative symmetric functions): \mathbf{h}_α and \mathbf{s}_α .

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- Tanisaki: $J_{(2,2)}$ is generated by e_1, e_2, e_3, e_4 , and

$$e_2(x_1, x_2, x_3), e_2(x_1, x_2, x_4), e_2(x_1, x_3, x_4), e_2(x_2, x_3, x_4), \\ e_3(x_1, x_2, x_3), e_3(x_1, x_2, x_4), e_3(x_1, x_3, x_4), e_3(x_2, x_3, x_4).$$

Theorem (Hotta-Springer, Garsia-Procesi)

The graded Frobenius characteristic of $R_\mu = \mathbb{C}[X]/J_\mu$ is the modified Hall-Littlewood symmetric function

$$\tilde{H}_\mu(x; t) = \sum_{\lambda} t^{n(\mu)} K_{\lambda\mu}(t^{-1}) s_\lambda$$

where $n(\mu) = \mu_2 + 2\mu_3 + 3\mu_4 + \dots$ and $K_{\lambda\mu}(t)$ is the Kostka-Foulkes polynomial.

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Corollary (Chevalley 1955)

The coinvariant algebra $\mathbb{C}[X]/(e_1, \dots, e_n)$ is isomorphic to the left regular representation of \mathfrak{S}_n .

$H_n(0)$ -action on polynomials

Definition

$H_n(0)$ acts on $\mathbb{C}[X]$ via the *Demazure operators*

$$\pi_i(f) = \frac{x_i f - x_{i+1} s_i f}{x_i - x_{i+1}}, \quad \forall f \in \mathbb{C}[X].$$

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Example

$$\begin{cases} \pi_1(x_1^3 x_2 x_3 x_4^4) = (x_1^3 x_2 + x_1^2 x_2^2 + x_1 x_2^3) x_3 x_4^4, \\ \pi_2(x_1^3 x_2 x_3 x_4^4) = x_1^3 x_2 x_3 x_4^4, \\ \pi_3(x_1^3 x_2 x_3 x_4^4) = x_1^3 x_2 (-x_3^2 x_4^3 - x_3^3 x_4^2). \end{cases}$$

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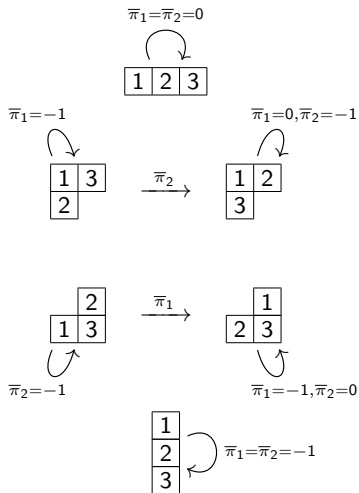
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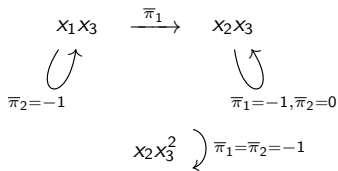
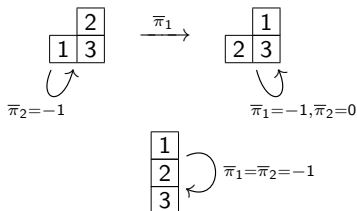
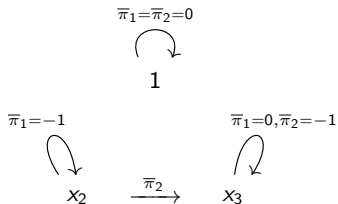
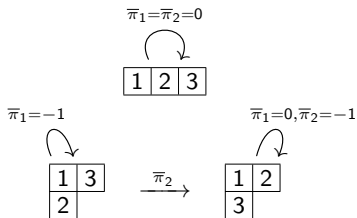
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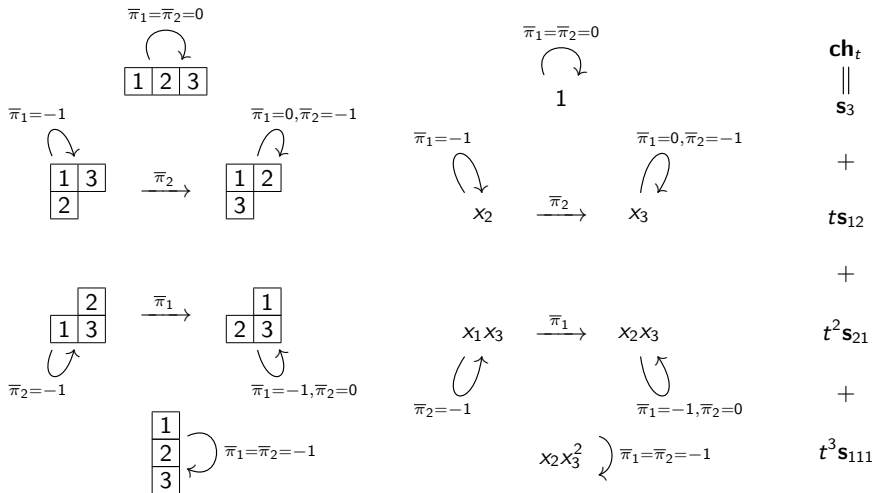
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- Representation theoretic interpretation?
- If α is a hook then R_{μ} still works.
- If α is not a hook then I need the Stanley-Reisner ring $\mathbb{C}[\mathcal{B}_n]$.

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- Then $R_\mu = \mathbb{C}[X]/J_\mu$ becomes a projective $H_n(0)$ -module.
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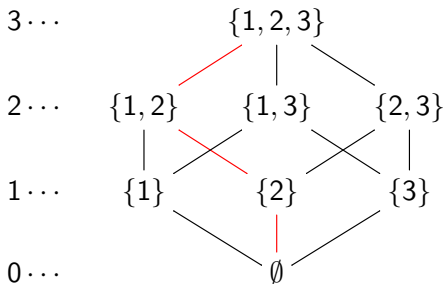
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- Its graded quasisymmetric characteristic is

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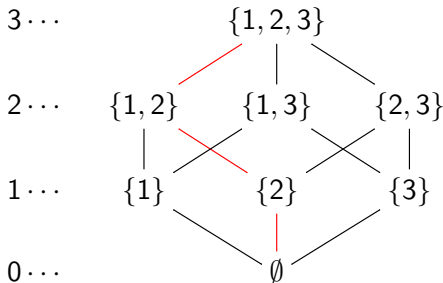
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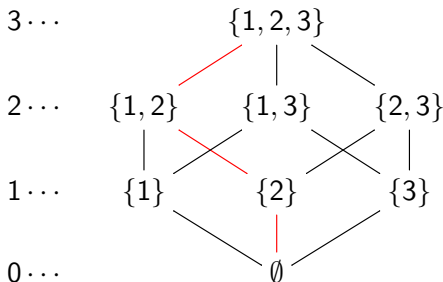
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- It has rank multiset $\{1, 2, 2, 3\}$.

- The Stanley-Reisner ring $\mathbb{C}[\mathcal{B}_n]$ of \mathcal{B}_n is the quotient of the polynomial algebra $\mathbb{C}[y_A : A \subseteq [n]]$ by the ideal

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- $y_{A_1} \cdots y_{A_k} \neq 0 \Leftrightarrow A_1, \dots, A_k$ form a multichain in \mathcal{B}_n .

- Basis: $\{y_M := y_{A_1} \cdots y_{A_k} : M = (A_1 \subseteq \cdots \subseteq A_k)\}$.

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- Transfer map $\tau(y_{2|1||3|}) = x_2(x_1 x_2)^2 x_1 x_2 x_3 = x_1^3 x_2^4 x_3$.
- The transfer map induces a vector space isomorphism

$$\tau : \mathbb{C}[\mathcal{B}_n]/(y_\emptyset) \cong \mathbb{C}[X]$$

(NOT a ring homomorphism: e.g. $y_1 y_2 = 0$ but $x_1 x_2 \neq 0$).

\mathfrak{S} -action on $\mathbb{C}[\mathcal{B}_n]$

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- Invariant algebra $\mathbb{C}[\mathcal{B}_n]^{\mathfrak{S}_n} = \mathbb{C}[\Theta]$, where $\Theta = \{\theta_0, \dots, \theta_n\}$.

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- Example:
$$\begin{cases} \bar{\pi}_1(y_{1|34||2|}) = y_{2|34||1|}, \\ \bar{\pi}_2(y_{1|34||2|}) = -y_{1|34||2|}, \\ \bar{\pi}_3(y_{1|34||2|}) = 0. \end{cases}$$

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- $\mathbb{C}[\mathcal{B}_n]/I_\alpha$ becomes a projective multigraded $H_n(0)$ -module.
- Its multigraded noncommutative characteristic is

$$\tilde{\mathbf{H}}_\alpha(\mathbf{x}; t_1, \dots, t_{n-1}) := \sum_{\beta \text{ refined by } \alpha} \underline{t}^{D(\beta)} \mathbf{s}_\beta$$

inside $\mathbf{NSym}[t_1, \dots, t_{n-1}]$, where $\underline{t}^S := \prod_{i \in S} t_i$.

Corollary (H.)

For any composition α of n one has

$$\tilde{\mathbf{H}}_{\alpha}(\mathbf{x}; t) = \tilde{\mathbf{H}}_{\alpha}(\mathbf{x}; t, t^2, \dots, t^{n-1}).$$

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For any composition α of n the (q, t) -analogue $\tilde{\mathbf{H}}_{\alpha}(\mathbf{x}; q, t)$ is a specialization of $\tilde{\mathbf{H}}_{1^n}(\mathbf{x}; t_1, \dots, t_{n-1})$:

$$t_i = \begin{cases} t^i, & i \in D(\alpha), \\ q^{n-i}, & i \in D(\alpha^c). \end{cases}$$

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Theorem (H.)

$$\begin{aligned}\mathrm{Ch}_{q,\underline{t}}(\mathbb{C}[\mathcal{B}_n]) &= \sum_{\alpha \models_0 n} \underline{t}^{D(\alpha)} \sum_{w \in \mathfrak{S}^\alpha} q^{\mathrm{inv}(w)} F_{D(w^{-1})} \\ &= \sum_{w \in \mathfrak{S}_n} \frac{q^{\mathrm{inv}(w)} \underline{t}^{D(w)} F_{D(w^{-1})}}{\prod_{0 \leq i \leq n} (1 - t_i)} \\ &= \sum_{k \geq 0} \sum_{\mathbf{p} \in [k+1]^n} t_{p'_1} \cdots t_{p'_k} q^{\mathrm{inv}(\mathbf{p})} F_{D(\mathbf{p})}.\end{aligned}$$

Corollary (Garsia and Gessel)

Applying $\sum_{\ell \geq 0} u_1^\ell \mathbf{ps}_{q_1; \ell+1}$ and the specialization $t_i = q_2^i u_2$ for all $i = 0, 1, \dots, n$ to the previous theorem, we obtain

$$\frac{\sum_{w \in \mathfrak{S}_n} q_0^{\text{inv}(w)} q_1^{\text{maj}(w^{-1})} u_1^{\text{des}(w^{-1})} q_2^{\text{maj}(w)} u_2^{\text{des}(w)}}{(u_1; q_1)_n (u_2; q_2)_n} = \sum_{\ell, k \geq 0} u_1^\ell u_2^k \sum_{(\lambda, \mu) \in B(\ell, k)} q_0^{\text{inv}(\mu)} q_1^{|\lambda|} q_2^{|\mu|}.$$

Here $(u; q)_n := (1 - u)(1 - uq) \cdots (1 - uq^n)$ and $\mathbf{ps}_{q; \ell}(F_\alpha) := F_\alpha(1, q, q^2, \dots, q^{\ell-1}, 0, 0, \dots)$.

Bipartite partitions

- A *bipartite partition* (λ, μ) satisfies
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$$\begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} 4 & 3 & 3 & 2 & 1 & 1 \\ 2 & 5 & 3 & 1 & 4 & 4 \end{pmatrix}$$
- $B(\ell, k)$ consists of bipartite partitions (λ, μ) such that $\max(\lambda) \leq \ell$ and $\max(\mu) \leq k$.

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Theorem (H.)

- (i) $\mathbb{C}(q)[\Delta(W)]^{H_W(q)} = \mathbb{C}(q)[\Theta]$.
- (ii) The $H_W(q)$ -action is Θ -linear.
- (iii) $\mathbb{C}(q)[\Delta(W)]/(\Theta) \cong H_W(q)$ if q is generic.

Questions for future research

- Define a nice $H_n(0)$ -action on the Stanley-Reisner ring of the Tits building $\Delta(G)$ of $G = GL(n, \mathbb{F}_q)$ (replacing multichains with multiflags).

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- Define a nice $H_n(0)$ -action on the Stanley-Reisner ring of the Tits building $\Delta(G)$ of $G = GL(n, \mathbb{F}_q)$ (replacing multichains with multiflags).
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- Find a character formula for each homogeneous component of $\mathbb{C}(q)[\mathcal{B}_n]/(\Theta)$ as an $H_n(q)$ -module; for $\mathbb{C}(q)[X]/(e_1, \dots, e_n)$ see Adin-Postnikov-Roichman.

Thank you!