0-Hecke algebra action on the Stanley-Reisner ring of the Boolean algebra

Jia Huang

University of Minnesota
⇓
University of Nebraska at Kearney

Jun 29, 2014

Outline

- The symmetric group and the 0-Hecke algebra.
- Actions on polynomials.
- Actions on Stanley-Reisner rings.
- Noncommutative Hall-Littlewood symmetric functions.
- Multivariate generating function of permutation statistics.

- The symmetric group $\mathfrak{S}_n$ is generated by the adjacent transpositions $s_i = (i, i+1)$, $1 \leq i \leq n-1$, with relations

$$\begin{cases} s_i^2 = 1, & 1 \leq i \leq n-1, \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, & 1 \leq i \leq n-2, \\ s_i s_j = s_j s_i, & |i-j| > 1. \end{cases}$$

The Symmetric Groups and 0-Hecke algebra

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- The 0-Hecke algebra $H_n(0)$ is generated by the bubble-sorting operators $\pi_1, \ldots, \pi_{n-1}$ with relations

$$\begin{cases} \pi_i^2 = \pi_i, & 1 \le i \le n-1, \\ \pi_i \pi_{i+1} \pi_i = \pi_{i+1} \pi_i \pi_{i+1}, & 1 \le i \le n-2, \\ \pi_i \pi_j = \pi_j \pi_i, & |i-j| > 1. \end{cases}$$

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- $$\begin{array}{rcl} \text{Representations of } \mathfrak{S}_n & \leftrightarrow & \text{symmetric functions,} \\ \text{direct sum} & \leftrightarrow & \text{sum,} \\ \text{induction product} & \leftrightarrow & \text{product.} \end{array}$$

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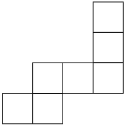
- 

$$\alpha = (2, 3, 1, 1) \models 7$$
$$D(\alpha) = \{2, 5, 6\} \subseteq [6]$$

Sym, QSym, and **NSym**

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- **NSym** (noncommutative symmetric functions): $\mathbf{h}_\alpha$ and $\mathbf{s}_\alpha$.

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- Tanisaki: $J_{(2,2)}$ is generated by e_1, e_2, e_3, e_4, and

$$e_2(x_1, x_2, x_3), e_2(x_1, x_2, x_4), e_2(x_1, x_3, x_4), e_2(x_2, x_3, x_4),$$
$$e_3(x_1, x_2, x_3), e_3(x_1, x_2, x_4), e_3(x_1, x_3, x_4), e_3(x_2, x_3, x_4).$$

R_μ as an $\mathfrak{S}_n$-representation

Theorem (Hotta-Springer, Garsia-Procesi)

The graded Frobenius characteristic of $R_\mu = \mathbb{C}[X]/J_\mu$ is the modified Hall-Littlewood symmetric function

$$\widetilde{H}_\mu(x;t) = \sum_\lambda t^{n(\mu)} K_{\lambda\mu}(t^{-1}) s_\lambda$$

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Corollary (Chevalley 1955)

The coinvariant algebra $\mathbb{C}[X]/(e_1, \ldots, e_n)$ is isomorphic to the left regular representation of $\mathfrak{S}_n$.

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Definition

$H_n(0)$ acts on $\mathbb{C}[X]$ via the *Demazure operators*

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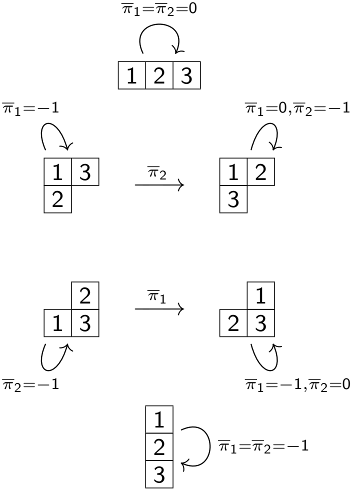

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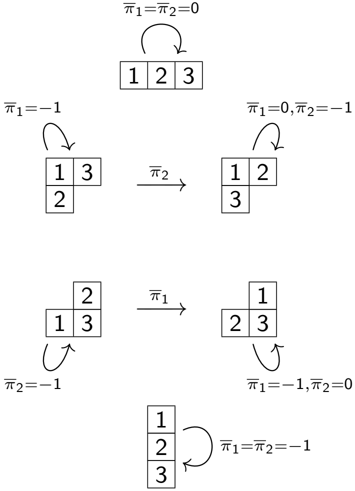

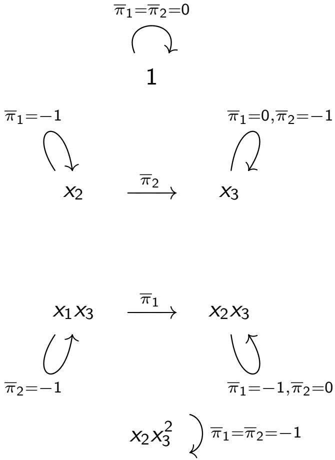

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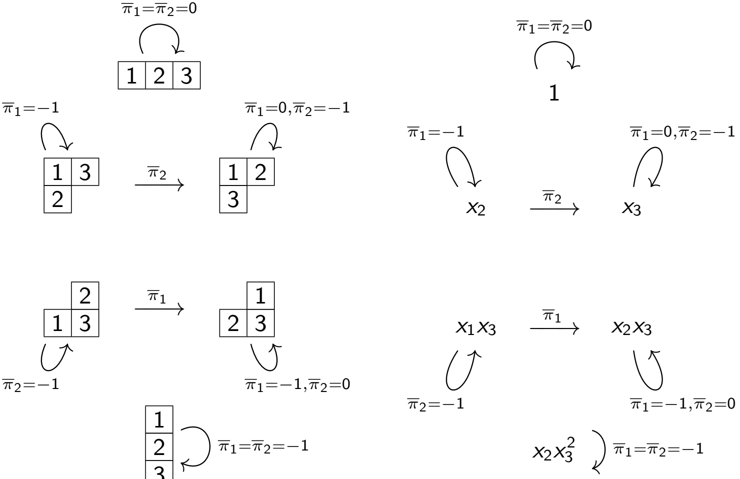

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- If α is a hook then R_μ still works.
- If α is not a hook then I need the Stanley-Reisner ring $\mathbb{C}[\mathcal{B}_n]$.

$H_n(0)$-action on $R_\mu = \mathbb{C}[X]/J_\mu$

Theorem (H. 2013)

- *The ideal J_μ is $H_n(0)$-stable if and only if $\mu = (1^k, n-k)$ is a hook. Assume μ is a hook below.*

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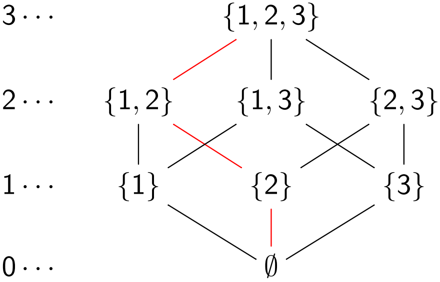
Boolean algebra

- The Boolean algebra $\mathcal{B}_n$ is the ranked poset of all subsets of $[n] := \{1, 2, \ldots, n\}$ ordered by inclusion.



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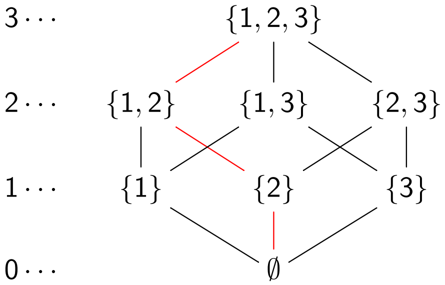

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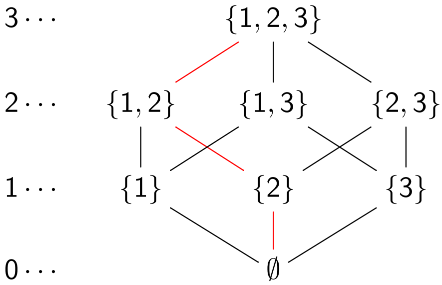

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- It has rank multiset $\{1, 2, 2, 3\}$.

- The Stanley-Reisner ring $\mathbb{C}[\mathcal{B}_n]$ of $\mathcal{B}_n$ is the quotient of

$$\text{the polynomial algebra } \mathbb{C}\left[y_A : A \subseteq [n]\right]$$

by the ideal

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- $y_{A_1} \cdots y_{A_k} \neq 0 \Leftrightarrow A_1, \ldots, A_k$ form a multichain in $\mathcal{B}_n$.

Properties

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- e.g. $y_{2|1||3|} = y_2 y_{12}^2 y_{123}$ has multigrading $\underline{t}^{1223} := t_1 t_2^2 t_3$.

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- Basis: $\{y_M := y_{A_1} \cdots y_{A_k} : M = (A_1 \subseteq \cdots \subseteq A_k)\}$.
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$$\tau : \mathbb{C}[\mathcal{B}_n]/(y_\emptyset) \cong \mathbb{C}[X]$$

(NOT a ring homomorphism: e.g. $y_1 y_2 = 0$ but $x_1 x_2 \neq 0$).

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- Invariant algebra $\mathbb{C}[\mathcal{B}_n]^{\mathfrak{S}_n} = \mathbb{C}[\Theta]$, where $\Theta = \{\theta_0, \ldots, \theta_n\}$.

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- We define an action of $H_n(0)$ on the Stanley-Reisner ring $\mathbb{C}[\mathcal{B}_n]$.

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- Example: $\begin{cases} \overline{\pi}_1(y_{1|34||2|}) = y_{2|34||1|}, \\ \overline{\pi}_2(y_{1|34||2|}) = -y_{1|34||2|}, \\ \overline{\pi}_3(y_{1|34||2|}) = 0. \end{cases}$

Theorem (H.)

- *$\mathbb{C}[\mathcal{B}_n]$ has a homogeneous $H_n(0)$-stable ideal I_α, $\forall \alpha \models n$.*

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- *$\mathbb{C}[\mathcal{B}_n]$ has a homogeneous $H_n(0)$-stable ideal I_α, $\forall \alpha \models n$.*
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- *$\mathbb{C}[\mathcal{B}_n]/I_\alpha$ becomes a projective multigraded $H_n(0)$-module.*
- *Its multigraded noncommutative characteristic is*

$$\widetilde{\mathbf{H}}_\alpha(\mathbf{x}; t_1, \ldots, t_{n-1}) := \sum_{\beta \text{ refined by } \alpha} \underline{t}^{D(\beta)} \mathbf{s}_\beta$$

inside $\mathbf{NSym}[t_1, \ldots, t_{n-1}]$, *where* $\underline{t}^S := \prod_{i \in S} t_i$.

Interpretation of noncommutative H-L functions

Corollary (H.)

For any composition α of n one has

$$\widetilde{\mathbf{H}}_\alpha(\mathbf{x};t) = \widetilde{\mathbf{H}}_\alpha(\mathbf{x};t,t^2,\ldots,t^{n-1}).$$

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For any composition α of n the (q,t)-analogue $\widetilde{\mathbf{H}}_\alpha(\mathbf{x};q,t)$ is a specialization of $\widetilde{\mathbf{H}}_{1^n}(\mathbf{x};t_1,\ldots,t_{n-1})$:

$$t_i = \begin{cases} t^i, & i \in D(\alpha), \\ q^{n-i}, & i \in D(\alpha^c). \end{cases}$$

- Let $\alpha \models n$. Define I_α to be the ideal of $\mathbb{C}[\mathcal{B}_n]$ generated by

$$\{\theta_i : i \in D(\alpha)\} \ \cup \ \{y_A : |A| \notin D(\alpha)\}.$$

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Quasisymmetric characteristic

Theorem (H.)

$$\begin{aligned}
\mathrm{Ch}_{q,\underline{t}}(\mathbb{C}[\mathcal{B}_n]) &= \sum_{\alpha \models_0 n} \underline{t}^{D(\alpha)} \sum_{w \in \mathfrak{S}^\alpha} q^{\mathrm{inv}(w)} F_{D(w^{-1})} \\
&= \sum_{w \in \mathfrak{S}_n} \frac{q^{\mathrm{inv}(w)} \underline{t}^{D(w)} F_{D(w^{-1})}}{\prod_{0 \le i \le n}(1 - t_i)} \\
&= \sum_{k \ge 0} \sum_{\mathbf{p} \in [k+1]^n} t_{p'_1} \cdots t_{p'_k} q^{\mathrm{inv}(\mathbf{p})} F_{D(\mathbf{p})}.
\end{aligned}$$

Applications to permutation statistics

Corollary (Garsia and Gessel)

Applying $\sum_{\ell \geq 0} u_1^\ell \mathbf{ps}_{q_1;\ell+1}$ *and the specialization* $t_i = q_2^i u_2$ *for all* $i = 0, 1, \ldots, n$ *to the previous theorem, we obtain*

$$\frac{\sum_{w \in \mathfrak{S}_n} q_0^{\mathrm{inv}(w)} q_1^{\mathrm{maj}(w^{-1})} u_1^{\mathrm{des}(w^{-1})} q_2^{\mathrm{maj}(w)} u_2^{\mathrm{des}(w)}}{(u_1; q_1)_n (u_2; q_2)_n} = \sum_{\ell, k \geq 0} u_1^\ell u_2^k \sum_{(\lambda,\mu) \in B(\ell,k)} q_0^{\mathrm{inv}(\mu)} q_1^{|\lambda|} q_2^{|\mu|}.$$

Here $(u; q)_n := (1-u)(1-uq)\cdots(1-uq^n)$ *and*

$$\mathbf{ps}_{q;\ell}(F_\alpha) := F_\alpha(1, q, q^2, \ldots, q^{\ell-1}, 0, 0, \ldots).$$

Bipartite partitions

- A *bipartite partition* (λ, μ) satisfies
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- For example, $\begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} 4 & 3 & 3 & 2 & 1 & 1 \\ 2 & 5 & 3 & 1 & 4 & 4 \end{pmatrix}$
- $B(\ell, k)$ consists of bipartite partitions (λ, μ) such that $\max(\lambda) \leq \ell$ and $\max(\mu) \leq k$.

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Theorem (H.)

(i) $\mathbb{C}(q)[\Delta(W)]^{H_W(q)} = \mathbb{C}(q)[\Theta]$.
(ii) The $H_W(q)$*-action is* Θ*-linear.*
(iii) $\mathbb{C}(q)[\Delta(W)]/(\Theta) \cong H_W(q)$ *if* q *is generic.*

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- Define a nice $H_n(0)$-action on the Stanley-Reisner ring of the Tits building $\Delta(G)$ of $G = GL(n, \mathbb{F}_q)$ (replacing multichains with multiflags).

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- Find a character formula for each homogeneous component of $\mathbb{C}(q)[\mathcal{B}_n]/(\Theta)$ as an $H_n(q)$-module; for $\mathbb{C}(q)[X]/(e_1, \ldots, e_n)$ see Adin-Postnikov-Roichman.

Thank you!